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Research Paper

Horizontal Shifting, a New Approach for Solving Polynomial Equations

Engineering

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Ismael Khorshed Abdulrahman

Assistant Lecturer, Erbil Polytechnic University, Erbil Technical Engineering College, Department of Information System Engineering., Erbil, Kurdistan Region, Iraq.

ABSTRACT

In this research, we describe a new method to solve specific polynomial equations. The method is different from any other earlier method and depends on shifting the polynomial equation horizontally on x-axis with a constant comes from the derivative of the function. By the study, a new general formula for shifting the polynomial equation is derived and proved. Also, a new simple formula is fixed for solving quadratic polynomial equations. Two specific cases for solving polynomial equations are studied with some examples.

1. Introduction:

A polynomial function is a function of the form:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x^1 + a_0$

Where:

$a_n, a_{n-1}, \ldots a_1, a_0$ are real numbers, $a_n \neq 0$

n is a positive integer number

The degree of a polynomial is the higher exponent of x appearing in a non-zero term of the polynomial. The root(s) of a polynomial are the value(s) of x which satisfy $f(x) = 0$. Graphs of polynomials are smooth and continuous. If n is an even integer, the graph looks like $f(x) = x^2$ but is wider and flatter near the origin, the higher the power, the flatter and stepper. If n is an odd integer, the graph looks similar to $f(x) = x^3$ but is wider and flatter near the origin, the higher the power, the flatter and stepper. Even degree polynomials rise on both left and right hand sides of the graph if the coefficient a_n is positive. The additional terms may cause the graph to have some turns near the center but will always have the same left and right hand behavior determined by the highest power term. Odd degree polynomials fall on the left and rise on the right hand sides of the graph if the coefficient a_n is positive. A polynomial of degree n can have at most (n-1) turning points. If we know the multiplicity of the zero(root), it tell us whether the graph crosses the x-axis at this point (odd multiplicities cross) or whether it just touches the axis and turns and heads back the other way (even multiplicities touch). [1]

2. Shifting of a polynomial function on x-axis (horizontal shifting)

The rule for shifting, stretching, compressing and reflecting the graph of a function are summarized in the following form [2]:

$y = af[b(x + c)] + d$

Where:

a: vertical stretch or compression; reflection about x-axis if negative

b: horizontal stretch or compression; reflection about y-axis if negative

c: horizontal shift

d: vertical shift

Applying the rule of horizontal shifting to f(x) by the constant (k):

$f(x + k) = a_n(x + k)^n + a_{n-1}(x + k)^{n-1} + a_{n-2}(x + k)^{n-2} + \cdots + a_2(x + k)^2 + a_1(x + k) + a_0$

$= a_n[\binom{n}{0}x^n + \binom{n}{1}kx^{n-1} + \binom{n}{2}k^2x^{n-2} + \cdots + \binom{n}{n-1}k^{n-1}x + \binom{n}{n}k^n] + a_{n-1}[\binom{n-1}{0}x^{n-1} + \binom{n-1}{1}kx^{n-2} + \binom{n-1}{2}k^2x^{n-3} + \cdots + \binom{n-1}{n-2}k^{n-2}x + \binom{n-1}{n-1}k^{n-1}] + a_{n-2}[\binom{n-2}{0}x^{n-2} + \binom{n-2}{1}kx^{n-3} + \binom{n-2}{2}k^2x^{n-4} + \cdots + \binom{n-2}{n-3}k^{n-3}x^1 + \binom{n-2}{n-2}k^{n-2}] + + \cdots + a_1(x + k) + a_0$

$f(x + k) = a_n(x^n + nkx^{n-1} + \left(\frac{n(n-1)}{2!}\right)k^2x^{n-2} + \cdots + \left(\frac{n(n-1)}{2!}\right)k^{n-2}x^2 + nk^{n-1}x + k^n) + a_{n-1}[x^{n-1} + (n-1)kx^{n-2} + (\frac{(n-1)(n-2)}{2!})k^2x^{n-3} + \cdots + (\frac{(n-1)(n-2)}{2!})k^{n-3}x^2 + (n-1)k^{n-2}x + k^{n-1}] + a_{n-2}(x^{n-2} + (n-2)kx^{n-3} + \left(\frac{(n-2)(n-3)}{2!}\right)k^2x^{n-4} + \cdots + \left(\frac{(n-2)(n-3)}{2!}\right)k^{n-4}x^2 + (n-2)k^{n-3}x + k^{n-2}) +$

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$a_2(x^2 + 2kx + k^2) + a_1(x + k) + a_0.$

$f(x + k) = a_nx^n + (na_nk + a_{n-1})x^{n-1} + [\left(\frac{n(n-1)}{2!}\right)k^2a_n + (n-1)ka_{n-1} + a_{n-2}]x^{n-2} + \ldots\ldots\ldots + \frac{1}{2!}[a_nn(n-1)k^{n-2} + a_{n-1}(n-1)(n-2)k^{n-3} + a_{n-2}(n-2)(n-3)k^{n-4} + \ldots\ldots + a_2]x^2 + \frac{1}{1!}[a_nnk^{n-1} + a_{n-1}(n-1)k^{n-2} + a_{n-2}(n-2)k^{n-3} + \ldots\ldots\ldots + a_1]x + \frac{1}{0!}(a_nk^n + a_{n-1}k^{n-1} + a_{n-2}k^{n-2} + \cdots\ldots + a_2k^2 + a_1k + a_0).$

$f(x + k) = a_nx^n + \left(\frac{1}{(n-1)!}\right)f^{(n-1)}(k)x^{n-1} + \left(\frac{1}{(n-2)!}\right)f^{(n-2)}(k)x^{n-2} + \cdots + \frac{1}{2!}f''(k)x^2 + \frac{1}{1!}f'(k)x + \frac{1}{0!}f(k).$

The benefit of shifting the polynomial equations appears when we want to eliminate the number of equation's terms. The new shifted formula $f(x+k)$ show us that there is a possibility to delete any term from the function f(x) except the first and the last terms. To delete any term fromf(x), we have to move f(x) on x-axis with a constant (k) equals to the value resulted from solving the n derivative of the function f(x).

For example, if we want to delete the second higher exponent term (i.e., x^{n-1}), we have to shift the f(x) by the constant (k) equals to $\left(\frac{-a_{n-1}}{2a_n}\right)$ because the function $f^{(n-1)}(x) = 0$ is a polynomial equation with degree one (linear equation) and it is solved by $x = \left(\frac{-a_{n-1}}{2a_n}\right)$.

To delete the third term of the function f(x) (i.e. x^{n-2} term), we have to shift f(x) with a constant equals to one root values of the derivative function $f^{(n-2)}(x) = 0$ which is a polynomial equation with degree two (quadratic equation) leading to give two zero values and so on.

The main use of deleting terms from the polynomial equation is that the solution of the original function f(x) will be easier.

3. Solution of quadratic equation:

$f(x) = ax^2 + bx + c$ (or $a_2x^2 + a_1x + a_0$)

$f'(x) = 2ax + b = 0$

$$\Rightarrow x = \frac{-b}{2a} = k$$

$$f'(k) = f'\left(\frac{-b}{2a}\right) = 0$$

Now we move f(x) on x-axis by the constant $\left(k = \frac{-b}{2a}\right)$ to get $f(x+k)$:

$$f(x+k) = ax^2 + f'(k)x + f(k) = 0$$

$$= ax^2 + f(k) = 0$$

$$\Rightarrow x = \pm\sqrt{\frac{-f(k)}{a}} \quad \text{(The solution of } f(x+k))$$

Now we return the shifted function $f(x+k)$ to the original position $f(x)$ by subtracting (k) from the solution as below:

$$x - k = \pm\sqrt{\frac{-f(k)}{a}}$$

$$x = k \pm\sqrt{\frac{-f(k)}{a}}$$

We can simplify the above equation by simplifying the amount under the square root as below:

$f(k) = ak^2 + bk + c$ ... (1)

$f'(k) = 2ak + b = 0 \Rightarrow b = -2ak$ ... (2)

Substitute eq. (2) in eq. (1) to get:

$$f(k) = ak^2 + (-2ak)k + c$$

$$f(k) = ak^2 - 2ak^2 + c$$

$$f(k) = -ak^2 + c$$

The amount with the square root $= \sqrt{\frac{-f(k)}{a}} = \sqrt{\frac{-(-ak^2+c)}{a}} = \sqrt{k^2 - \frac{c}{a}}$

Hence the solution of $f(x)$ becomes:

$x = k \pm \sqrt{k^2 - \frac{c}{a}}$, where k is found by the solution of $f'(x) = 0$

The general solution of the quadratic equation can be derived from this formula:

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b = 0$$

$$\Rightarrow x = \frac{-b}{2a} = k$$

$$f(k) = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$$

$$f(k) = a\left(\frac{b^2}{4a^2}\right) + \left(\frac{-b^2}{2a}\right) + c$$

$$f(k) = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$f(k) = \frac{-(b^2 - 4ac)}{4a}$$

But $x = k \pm \sqrt{\frac{-f(k)}{a}}$ and $k = \frac{-b}{2a}$

$$x = \frac{-b}{2a} \pm \sqrt{\frac{(b^2-4ac)}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{(b^2-4ac)}}{2a}$$

Example:

$x^2 - 2x + 2 = 0$

$2x - 2 = 0 \Rightarrow x = 1 = k$

$x = 1 \pm \sqrt{1-2} = \{1 \pm i\}$

There are two another uses of the shifting property of polynomial equations.

First:

If the polynomial roots are the same, then all the terms after shifting to (k) are set to zero except the first and the last. The shifting formula is very useful to check and solve polynomial equations with this type applied to any value of (n). If the degree of the equation is odd, then the solution is unique value and hence:

$x = k + \sqrt[n]{f(k)}$

While if the degree is even, then there are two solution of the equation:

$x = k \pm \sqrt[n]{f(k)}$

Let us take an example:

Example:

Solve $f(x) = 2x^3 - \sqrt{6}x^2 + x - 1 = 0$

Solution:

$f'(x) = 6x^2 - 2\sqrt{6}x + 1$

$f''(x) = 12x - 2\sqrt{6} = 0 \Rightarrow x = \frac{1}{\sqrt{6}} = k$

$f'(\frac{1}{\sqrt{6}}) = 0$

$f(\frac{1}{\sqrt{6}}) = \frac{1-3\sqrt{6}}{3\sqrt{6}}$

$f\left(x + \frac{1}{\sqrt{6}}\right) = 2x^3 + \frac{1-3\sqrt{6}}{3\sqrt{6}} = 0$

$x = \frac{1}{\sqrt{6}} + \sqrt[3]{\frac{3\sqrt{6}-1}{6\sqrt{6}}}$

The solution of $f(x)$ has unique real value. The other two roots are complex number and can be determined by long division or by solving the quadratic equation explained below:

$ax^2 + (ax_1 + b)x + (ax_1^2 + bx_1 + c) = 0$

$where\ x_1\ is\ the\ first\ determined\ root$

The radical method used for solving cubic polynomial equations is not easy and needs too complex calculations to reach to the solution.

Second:

If all the neighbor roots of any polynomial equation have the same distance between them, then all the even terms (including the absolute value) will be deleted from the shifted equation if (n) is odd, and all the odd terms will be deleted if (n) is even.

Example1:

$f(x) = x^3 - 3.3x^2 + 1.38x + 1.144$

$f'(x) = 3x^2 - 6.6x + 1.38$

$f''(x) = 6x - 6.6 = 0 \Rightarrow x = 1.1 = k$

$f(1.1) = 0, f'(1.1) = 2.25$

$f(x + 1.1) = x^3 - 2.25x = 0$

$x(x^2 - 2.25) = 0$

$\{x = 0\} + 1.1 = \{1.1\}$

$\{x = \pm\sqrt{2.25} = \pm 1.5\} + 1.1 = \{2.6, -0.4\}$

Hence the roots are: {-0.4, 1.1, 2.6}

It is clear from the solution that the distance between the first and the second roots is equal to the distance between the second and the third roots and equals to 1.5. However, the solution of the above equation is much complicate without using the shifting property.

Example2:

$f(x) = x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

$f'(x) = 4x^3 - 48x^2 + 172x - 176$

$f''(x) = 12x^2 - 96x + 172$

$f'''(x) = 24x - 96 = 0 \Rightarrow x = 4 = k$

$f(4) = 9, f'(4) = 0, f''(4) = -20$

$f(x + 4) = x^4 + \frac{1}{2!}(-20)x^2 + 9 = 0$

$(x^2)^2 - 10x^2 - 9 = 0$

$\{x^2 = 5 \pm \sqrt{25 - 9}\} + 4 = \{1, 9\}$

$x = \{\pm 1, \pm 3\} + 4 = \{1, 3, 5, 7\}$

There is no solution by the radical for degree n ≥ 5 while it is so easy by the method of the shifting property for the cases discussed in this paper.

This special case of polynomial equation is more close to the pattern of series expansion of the sine and cosine functions.

$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n + \frac{x^{2n}}{(2n)!}$

$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n + \frac{x^{2n+1}}{(2n+1)!}$

The sine and cosine curves intersect the x-axis periodically and regulatory with equal distance between any two neighbor roots ($distance = \pi$). They are similar to polynomial equation with the second type took about. So, according to the shifting property – second note, the cosine function should be free of any odd terms while the sine function must be free of even terms. Because the domain is infinity, so the degree of polynomial equation will go to infinity also (n degree).

4. The solution of the cubic function

4.1 Introduction:

The shifting formula of a polynomial equation leads to a significant approach of solving the cubic functions. As a literature review, it is known that the standard method of solving the cubic polynomial equation, generally known Cardan's solution is:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+\sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}-\frac{b}{3a}$$

This formula subjected to some modifications during many years ago. The more recent modification is that one which is done by the reference (1). The approach of this modification describes five fundamental parameters of the cube (δ, γ, h, x_N *and* y_N). Where $\delta=\sqrt{\frac{b^2-3ac}{9a^2}}$, $h=2a\delta^3$, $\cos 3\theta=\frac{-y_N}{h}$ and $y_N=\frac{2b^3}{27a^2}-\frac{bc}{3a}+d$

And the roots (α, β *and* γ) are:

$$\alpha=x_N+2\delta\cos\theta$$

$$=x_N+2\sqrt{\frac{b^2-3ac}{9a^2}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-\left(\frac{2b^3}{27a^2}-\frac{bc}{3a}+d\right)}{2a\left(\frac{b^2-3ac}{9a^2}\right)^{\frac{3}{2}}}\right)\right)$$

$$\beta=x_N+2\delta\cos\left(\theta+\frac{2\pi}{3}\right)$$

$$=x_N+2\sqrt{\frac{b^2-3ac}{9a^2}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-\left(\frac{2b^3}{27a^2}-\frac{bc}{3a}+d\right)}{2a\left(\frac{b^2-3ac}{9a^2}\right)^{\frac{3}{2}}}\right)+\frac{2\pi}{3}\right)$$

$$\gamma=x_N+2\delta\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$=x_N+2\sqrt{\frac{b^2-3ac}{9a^2}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-\left(\frac{2b^3}{27a^2}-\frac{bc}{3a}+d\right)}{2a\left(\frac{b^2-3ac}{9a^2}\right)^{\frac{3}{2}}}\right)+\frac{4\pi}{3}\right)$$

The above standard Cardan's solution and the modified revealed methods depend on the coefficients (a, b, c, d) of the origin cubic. These formulas are not easy to be memorized in our mind and need more calculations as it is clear from their forms. While the new approach we are discussing here in this paper considers a different procedures lead to the roots of the cubic easily. The roots of complex numbers and the rectangular-to-polar conversion are utilized in this approach for solving the equations.

4. 2 The new approach

Assume that the origin function of the cubic is:

$$f(x)=x^3+bx^2+cx+d=0 \qquad \dots eq.(1)$$

Where the first coefficient $a=+1$. If $a\neq 1$, then we divide all the equation by (a).

The first step we start to solve the $eq.(1)$ is to find the derivatives $f'(x)$ and $f''(x)$, and make $f''(x)=0$ to find (k) then $f'(k)$ and $f(k)$ which are needed to write the shifted formula $f(x+k)$:

$$f(x+k)=x^3+\frac{f''(k)}{2}x^2+f'(k)x+f(k) \qquad \dots eq.(2)$$

But $f''(k)=0$, therefore, $eq.(2)$ will become:

$$f(x+k)=x^3+f'(k)x+f(k) \qquad \dots eq.(3)$$

Now, the solution of $f(x+k)$ will be the value of (Z):

$$Z=\sqrt[3]{-4f(k)+\sqrt{\left(4f(k)\right)^2+\left(\frac{4}{3}f'(k)\right)^3}} \quad \dots eq.(4)$$

or $$Z=\sqrt[3]{-4D+\sqrt{(4D)^2+\left(\frac{4}{3}C\right)^3}} \quad \dots eq.(5)$$

Where $D=f(k)$ and $C=f'(k)$.

Due to the square root in $eq.(5)$, the value of Z may be complex if $(4D)^2+\left(\frac{4}{3}C\right)^3<0$ and this leads to three real roots of the equation. Or Z may be real number if $(4D)^2+\left(\frac{4}{3}C\right)^3>0$ which will lead to one real root and two conjugate imaginary roots. Or the solution has three real equal roots if $(4D)^2+\left(\frac{4}{3}C\right)^3=0$. It is important to mention that if the sign of $C(=f'(k))$ is positive, then the roots will be one real and two imaginary roots because Z will be real number in this case.

In the case of complex value of Z, we determine the polar representation of Z and taking its real value as the solution has three real roots.

$$Z=Re\sqrt[3]{-4D+\sqrt{(4D)^2+\left(\frac{4}{3}C\right)^3}}$$

$$Z=Re\sqrt[3]{re^{i\theta}}=Re\sqrt[3]{r}e^{i\left(\frac{\theta}{3}+\frac{2n\pi}{3}\right)}=\sqrt[3]{r}\cos\left(\frac{\theta}{3}+\frac{2n\pi}{3}\right)$$

$\dots eq.(7)$

Where (r) is the magnitude of the the complex (Z) and (θ) is the angle of (Z) while (n) takes the values 0,1,2 for the three roots.

All the three roots of $re^{i\theta}$ lie on a circle centered at the origin and having radius equal to the real, positive third root of r ($\sqrt[3]{r}$ or $|Z|$). One of them has argument $\left(\frac{\theta}{3}\right)$, the others are uni-formly spaced around the circle, each being separated from its neighbors by an angle equal to $\left(\frac{2\pi}{3}\right)$.

Figure 1 illustrates the placement of the three roots, w_0, w_1, w_2, of the complex number $Z = re^{i\theta}$

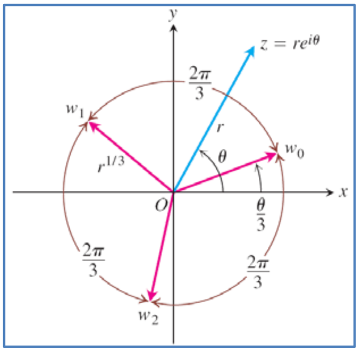

Figure 1The three cube roots of Z=re^iθ

Finally, the solution of the origin cubic $f(x)$ will be the solution of the shifted cubic $f(x+k)$ plus the shifting value (k).

$\therefore$ Roots of $f(x)= w_0, w_1, w_2 = Re\left(r^{\frac{1}{3}}e^{i\left(\frac{\theta}{3}+\frac{2n\pi}{3}\right)}\right) + k$

$$w_0 = \left(r^{1/3}\cos\frac{\theta}{3}\right) + k$$

$$w_1 = \left(r^{1/3}\cos\frac{\theta}{3} + \frac{2\pi}{3}\right) + k$$

$$w_2 = \left(r^{\frac{1}{3}}\cos\frac{\theta}{3} + \frac{4\pi}{3}\right) + k$$

In the case of real value of Z, we find the roots from $eq.(5)$ by extending the range of (θ) to include complex angles, then all cubics can be solved using this approach. Another approach, is finding the single real root directly from the relation $\frac{Z^2-C}{6Z}$. The rest of the roots can be found by the long division.

Example: solve the equation

$$f(x) = x^3 + 3x^2 - 2x - 1 = 0$$

Solution:

$$f'(x) = 3x^2 + 6x - 2$$

$$f''(x) = 6x + 6 = 0 \qquad \Rightarrow x = -1 \quad (\therefore k = -1)$$

$$f'(-1) = -5 \quad \& \quad f(-1) = 3$$

$$f(x+k) = x^3 - 5x + 3 = 0$$

$$\text{roots} = \text{Re}\sqrt[3]{-4*3 + \sqrt{(4*3)^2 + \left(\frac{4}{3}(-5)\right)^3}} - 1$$

$$= \text{Re}\sqrt[3]{-12 + i12.340838} - 1$$

$$= \text{Re}\sqrt[3]{17.213259e^{i\left(134.2°*\frac{\pi}{180°}\right)}} - 1$$

$$= 2.5819888 * \cos\left(\frac{134.2°}{3} + (0, \frac{2\pi}{3}, \frac{4\pi}{3}\right)$$

$$w_0 = 1.834243185 - 1 = 0.834243185$$

$$w_1 = -2.490863615 - 1 = -3.490863615$$

$$w_2 = 0.65662043 - 1 = -0.34337957$$

Conclusion

Eliminating the number of terms of a polynomial function simplify the solution of its equation. The shifting property is a useful tool for this elimination. Also, it is a tool that can be used for checking the type of the roots whether the equation has a unique root or not, or whether the equation intersects the x-axis regulatory or not and find these roots. The solution of these equations by the radical is too difficult (or impossible if n ≥ 5) while it is easy by the introduced method. From the proved examples shown in this paper, it can be seen that the solution of quadratic polynomial equation is too easy by the presented method. It takes its advantages from the derivative and shifting methods. Also, the steps for solving cubic equation can be decreased in little easy procedure. In some special cases, it can be solved just by shifting the function on x-axis with the derivates value. The shifting property can be applied on any degree of the polynomial equation; hence, it exceeds any other method for solving these types of equations in a fast.

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