Volume : 3 | Issue : 1 | January 2014 • ISSN No 2277 - 8179

Research Paper

On uniform continuity of $x^m \sin(1/x^n)$

Mathematics

KEYWORDS : Continuity , Uniform continuity

Neel Kumari	Govt. Teacher,Government of Haryana,India
Yogesh Kumar	Research Scholar,Department of mathematics IIT Delhi, India

ABSTRACT

In this paper we will find conditions on n,m∈N so that the function x^msin(1/x^n) become uniformly continuous on the domain (0,∞).

1 INTRODUCTION

In this paper we will find conditions on n,m∈ N so that the function $x^m \sin(\frac{1}{x^n})$ become uniformly continuous on the domain $(0,\infty)$. We will use many results and theorems for the final result. We will state and prove the required results from elementary real analysis.

There are several applications of continuity and of uniform continuity in every branch of sciences as well as in our daily life uses.

2 PRELIMINARIES

Definition 1.1 Let f be a real valued function defined on a set $S \subseteq \mathbb{R}$. Then f is said to be uniformly continuous on S if

For every $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$ and $|x-y| < \delta$ imply $|f(x) - fy<\varepsilon$

Theorem 1.1[1] A real valued function f on (a,b) is uniformly continuous on (a,b) if and only if it can be extended to a continuous function $\tilde{f}$ on [a,b]

Theorem 1.2[1] Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing the end points of I . If f is differentiable on I° and if $f^{'}$(x) is bounded on I°,then f is uniformly continuous on I.

Result 1.1[3] Suppose that f : $[1,\infty) \to \mathbb{R}$ is uniformly continuous. Then there is a positive M such that $\frac{|f(x)|}{x} \leq M$ for $x \geq 1$.

Proof: By unform continuity of f on $[1,\infty)$ there exists $\delta > 0$ such that $|f(x) - f(\check{x})| < 1$ if $|x - \check{x}| \leq \delta$. Any $x \geq 1$ can be written in the form $x = 1 + n\delta + r$, where $n \in \mathbb{N} \cup \{0\}$ and $0 \leq r < \delta$. Hence

$$|f(x)| \leq |f(1)| + |f(x) - f(1)|$$

$$\leq |f(1)| + (n+1)$$

Dividing by x gives

$$\frac{|f(x)|}{x} \leq \frac{|f(1)|+n+1}{1+n\delta+r} \leq \frac{|f(1)|+2}{\delta} = M$$

Result 1.2 If a function f: $(0,\infty) \to \mathbb{R}$ be continuous and satisfies the condition that if

$\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$ are finite reals then the function f is bounded on the interval $(0,\infty)$.

Proof : Since $\lim_{x\to\infty} f(x)$ is finite say l then by definition of limit for each $\epsilon > 0$ there exists a real number say k such that $f(x) \in (l-\epsilon, l+\epsilon)$ for all $x \geq k$. That is function f is bounder on the interval $[k,\infty)$. Again since $\lim_{x\to 0} f(x)$ is a finite real number so we can extent the function continuously on the interval $[0,k]$ and hence the function is bounded on the the interval$[0,k]$. In either way we can say that the function f is bounder on the interval$(0,\infty)$.

Result 1.3[3] If f : $[a,\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x)$ is finite, then f is uniformly continuous on $[a,\infty)$.

3.MAIN RESULTS.

For $\boldsymbol{m = n}$ the function $x^m \sin(\frac{1}{x^n}) = x^n \sin(\frac{1}{x^n}) = \frac{\sin(\frac{1}{x^n})}{\frac{1}{x^n}}$ and $\lim_{x\to\infty} \frac{\sin(\frac{1}{x^n})}{\frac{1}{x^n}}$ =1 so the function $x^m \sin(\frac{1}{x^n})$ is uniformly continuous on the interval $[1,\infty)$ again since $\lim_{x\to 0} x^m \sin(\frac{1}{x^n})$ =0 so the

Volume : 3 | Issue : 1 | January 2014 • ISSN No 2277 - 8179

fuction can be extended continuously on the interval [0,1] and hence uniformly continuous on the the interval [0.1] as well as on the interval $[1,\infty)$. So the function $x^m \sin(\frac{1}{x^n})$ is uniformly continuous on the interval $(0,\infty)$ by result 1.3.

If **m=n+1** then the function f(x)= $x^m \sin(\frac{1}{x^n})$ becomes $x^{n+1}\sin(\frac{1}{x^n})$ and

f'(x) $= (n+1)x^n \sin(\frac{1}{x^n}) - cos\,\frac{1}{x^n}$

Again since $\lim_{x\to\infty} f'$(x) $=$ n so the derivative of the function f that is f'(x) is bounded on the interval $[1,\infty)$ so the function is uniformly continuous on the interval$[1,\infty)$ by theorem 1.3. Again function f is also uniformly continuous on (0,1] as can be extended continuously on the interval [0,1] and hence f is uniformly continuous on the interval $(0,\infty)$ if $m = n+1$.

If $\boldsymbol{m} < n$ then the function f(x) $=$ $x^m \sin(\frac{1}{x^n})$ $=$ $\frac{1}{x^r}\frac{\sin(\frac{1}{x^n})}{\frac{1}{x^n}}$ where r is such that $m + r = n$. Now in this case $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = 0$ and hence the function is uniformly continuous in this by theorem 1.1.

If $\boldsymbol{m} > n+1$ then $\frac{|f(x)|}{x} = \frac{|x^m \sin\left(\frac{1}{x^n}\right)|}{x} = |\frac{x^m \sin\left(\frac{1}{x^n}\right)}{x}| = |x^{m-1}\sin(\frac{1}{x^n})\,|$ on the interval $[1,\infty)$.

But since $\lim_{x\to\infty} x^{m-1}\sin\left(\frac{1}{x^n}\right) = \lim_{x\to\infty} x^r \frac{\sin\left(\frac{1}{x^n}\right)}{\frac{1}{x^n}} \to \infty$ as $m > n+1$ and hence there does not exist any positive number M such that $\frac{|f(x)|}{x} \leq M$ for $x \geq 1$. Threre for by result f is not uniformly continuous on the interval $(0,\infty)$.

4.CONCLUSION.

By above discussion we conclude that the function f(x)= $x^m \sin(\frac{1}{x^n})$ defined on the interval $(0,\infty)$ is uniformly continuous if $m = n\ or\ m = n+1$ for $m > n+1$ it is not uniformly continuous on $(0,\infty)$

REFERENCE

[1]Ross K.A. (2010), "Elementary Analysis: The theory of calculus" Springer(2000). | [2]Rudin W.,"Principles of Mathematical analysis." McRRAW-HIL INTERNATIONAL EDITION(1976). | [3]Kaczor W.J. and Nowak M.T.203, (2003), "Problems in Mathematical Analysis 2 Continuity and Differentation." AMS STUDENT MATHEMATICAL LIBRARY VOLUME 12(2000). |