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# Pell Trigonometry

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**ABSTRACT**

*In this paper, I have derived trigonometric relations involving Pell and Pell-Lucas numbers using their properties.*

## 1. Introduction

Pell and Pell-Lucas numbers are respectively defined by the recurrence relation

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2,\ Q_1 = 2. \end{cases} \quad (1)$$

Their Binet formulae are $P_n = \dfrac{\gamma^n - \delta^n}{\gamma - \delta}$

and $Q_n = \gamma^n + \delta^n$, where γ and δ are the roots of $x^2 - 2x - 1 = 0$ i.e. $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ so that $\gamma\delta = -1$ or $\delta = \dfrac{-1}{\gamma}$ or $\gamma = \dfrac{-1}{\delta}$.

## 2. Pell Trigonometric Identities

**Theorem 1:**

$$\tan\left(\tan^{-1}\frac{P_n}{P_{n+1}} - \tan^{-1}\frac{P_{n+1}}{P_{n+2}}\right) = \frac{(-1)^{n+1}}{2P_{2n+2}}.$$

**Proof:**

$$\text{L.H.S.} = \tan\left(\tan^{-1}\frac{P_n}{P_{n+1}} - \tan^{-1}\frac{P_{n+1}}{P_{n+2}}\right)$$

$$= \tan\left[\tan^{-1}\frac{\left(\frac{P_n}{P_{n+1}} - \frac{P_{n+1}}{P_{n+2}}\right)}{1 + \left(\frac{P_n}{P_{n+1}}\right)\left(\frac{P_{n+1}}{P_{n+2}}\right)}\right]$$

$$= \frac{\left(\frac{P_n}{P_{n+1}} - \frac{P_{n+1}}{P_{n+2}}\right)}{1 + \left(\frac{P_n}{P_{n+1}}\right)\left(\frac{P_{n+1}}{P_{n+2}}\right)}$$

√ $$= \frac{P_n P_{n+2} - P_{n+1}^2}{P_{n+1}P_{n+2} - P_{n+1}P_n} = \frac{(-1)^{n+1}}{2P_{2n+2}}.$$

$$\left(\begin{array}{l} \because\ P_n P_{n+2} - P_{n+1}^2 = (-1)^{n+1} \\ and\ P_{n+1}P_{n+2} - P_{n+1}P_n = 2P_{2n+2} \end{array}\right)$$

**Theorem 2:**

$$\tan\left(\tan^{-1}\frac{Q_n}{Q_{n+1}} - \tan^{-1}\frac{Q_{n+1}}{Q_{n+2}}\right) = \frac{8(-1)^{n+1}}{Q_{2n+3} + Q_{2n+1}}.$$

**Proof:**

$$\text{L.H.S.} = \tan\left(\tan^{-1}\frac{Q_n}{Q_{n+1}} - \tan^{-1}\frac{Q_{n+1}}{Q_{n+2}}\right)$$

$$= \tan\left[\tan^{-1}\frac{\left(\frac{Q_n}{Q_{n+1}}-\frac{Q_{n+1}}{Q_{n+2}}\right)}{1+\left(\frac{Q_n}{Q_{n+1}}\right)\left(\frac{Q_{n+1}}{Q_{n+2}}\right)}\right]$$

$$=\frac{\left(\frac{Q_n}{Q_{n+1}}-\frac{Q_{n+1}}{Q_{n+2}}\right)}{1+\left(\frac{Q_n}{Q_{n+1}}\right)\left(\frac{Q_{n+1}}{Q_{n+2}}\right)}$$

$$=\frac{Q_nQ_{n+2}-Q_{n+1}^2}{Q_{n+1}Q_{n+2}-Q_{n+1}Q_n}$$

$$=\frac{8(-1)^{n+1}}{Q_{2n+3}+2(-1)^{n+1}+Q_{2n+1}+2(-1)^n}.$$

$$\left(\begin{array}{l}\because\ Q_nQ_{n+2}-Q_{n+1}^2=8(-1)^n\\ and\ Q_nQ_{n+1}=Q_{2n+1}+2(-1)^n\end{array}\right)$$

$$=\frac{8(-1)^{n+1}}{Q_{2n+3}+Q_{2n+1}}.$$

**Theorem 3:**

$$\tan^{-1}\frac{P_n}{P_{n+1}}=\sum_{i=1}^{n}(-1)^{i+1}\tan^{-1}\frac{1}{P_{2i}}.$$

**Proof:** (By PMI).

When $n = 1$, L.H.S. = $\tan^{-1}\frac{P_1}{P_2}=\tan^{-1}\frac{1}{2}.$

R.H.S. = $(-1)^2\tan^{-1}\frac{1}{P_2}=\tan^{-1}\frac{1}{2},$

$\therefore$ L.H.S. = R.H.S.

Therefore the formula works for $n = 1$. Now, assume that it works for $n = k$, then

$$\sum_{i=1}^{k+1}(-1)^{i+1}\tan^{-1}\frac{1}{P_{2i}}=$$

$$=\sum_{i=1}^{k}(-1)^{i+1}\tan^{-1}\frac{1}{P_{2i}}+(-1)^{k+2}\tan^{-1}\frac{1}{P_{2(k+1)}}$$

$$=\tan^{-1}\frac{P_k}{P_{k+1}}+(-1)^k\tan^{-1}\frac{1}{P_{2(k+1)}}$$

$$=\tan^{-1}\frac{(-1)^{k+1}}{P_{2k+2}}+\tan^{-1}\frac{P_{k+1}}{P_{k+2}}+(-1)^k\tan^{-1}\frac{1}{P_{2k+2}}$$

$$\left[\because \tan\left(\tan^{-1}\frac{P_n}{P_{n+1}}-\tan^{-1}\frac{P_{n+1}}{P_{n+2}}\right)=\frac{(-1)^{n+1}}{2P_{2n+2}}\right]$$

$$=\tan^{-1}\frac{P_{k+1}}{P_{k+2}}\quad\left[\because\ \tan^{-1}(-x)=-\tan^{-1}(x)\right]$$

So the formula works for $n = k + 1$. Thus by PMI, the formula holds for every $n\geq 1$.

**Corollary 1:** $\sum_{n=1}^{\infty}(-1)^{n+1}\tan^{-1}\frac{1}{P_{2n}}=\tan^{-1}\frac{1}{\gamma}.$

**Proof:** Since $\tan^{-1}x$ is a continuous increasing function, $\tan^{-1}\frac{1}{P_{2n}}>\tan^{-1}\frac{1}{P_{2n+2}}.$

Also $\lim_{n\to\infty}\tan^{-1}\frac{1}{P_{2n}}=\tan^{-1}0=0$, therefore the series converges and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\tan^{-1}\frac{1}{P_{2n}}$$

$$=\lim_{m\to\infty}\sum_{n=1}^{m}(-1)^{n+1}\tan^{-1}\frac{1}{P_{2n}}$$

$$=\lim_{m\to\infty}\tan^{-1}\frac{P_m}{P_{m+1}}$$

$$=\tan^{-1}\left(\lim_{m\to\infty}\frac{P_m}{P_{m+1}}\right)$$

**Theorem 4:**

$$\tan^{-1}\frac{2}{P_{2n+1}} = \tan^{-1}\frac{1}{P_{2n}} - \tan^{-1}\frac{1}{P_{2n+2}}. \quad (2)$$

**Proof:** Let $\theta_n = \tan^{-1}\frac{1}{P_{2n}} - \tan^{-1}\frac{1}{P_{2n+2}}$,

then

$$\tan\theta_n = \frac{\frac{1}{P_{2n}} - \frac{1}{P_{2n+2}}}{1+\frac{1}{P_{2n}}\times\frac{1}{P_{2n+2}}} = \frac{P_{2n+2}-P_{2n}}{P_{2n}P_{2n+2}+1}$$

$$= \frac{2P_{2n+1}}{P_{2n+1}^2}$$

$$\left(\begin{array}{l} \because \quad P_nP_{n+2} - P_{n+1}^2 = (-1)^{n+1} \\ and\ P_n - P_{n-2} = 2P_{n-1} \end{array}\right)$$

$$= \frac{2}{P_{2n+1}}$$

Hence, $\tan^{-1}\frac{2}{P_{2n+1}} = \tan^{-1}\frac{1}{P_{2n}} - \tan^{-1}\frac{1}{P_{2n+2}}$.

**Theorem 5:** $\tan^{-1}\left(\frac{4\sqrt{2}}{2P_{2n+1}-P_{2n+1}^{-1}}\right)$

$$= \tan^{-1}\left(\frac{2\sqrt{2}}{Q_{2n}}\right) + \tan^{-1}\left(\frac{2\sqrt{2}}{Q_{2n+2}}\right).$$

**Proof:** Let $\theta_n = \tan^{-1}\left(\frac{2\sqrt{2}}{Q_{2n}}\right) + \tan^{-1}\left(\frac{2\sqrt{2}}{Q_{2n+2}}\right)$,

then

$$\tan\theta_n = \frac{\frac{2\sqrt{2}}{Q_{2n}} - \frac{2\sqrt{2}}{Q_{2n+2}}}{1+\frac{2\sqrt{2}}{Q_{2n}}\times\frac{2\sqrt{2}}{Q_{2n+2}}} = \frac{2\sqrt{2}\left(Q_{2n+2}+Q_{2n}\right)}{Q_{2n}Q_{2n+2}-8}$$

$$= \frac{2\sqrt{2}\left(8P_{2n+1}\right)}{Q_{2n+1}^2}\left(\begin{array}{l} \because \quad Q_nQ_{n+2} - Q_{n+1}^2 = 8(-1)^n \\ and\ Q_{n-1} + Q_{n+1} = 8P_n \end{array}\right)$$

$$= \tan^{-1}\frac{1}{\gamma}.$$

$$= \frac{16\sqrt{2}P_{2n+1}}{8P_{2n+1}^2 - 4}$$

$$\left(\because \quad 8P_n^2 = Q_n^2 - 4(-1)^n\right)$$

$$= \frac{4\sqrt{2}}{2P_{2n+1} - P_{2n+1}^{-1}}.$$

Hence, $\tan^{-1}\left(\frac{4\sqrt{2}}{2P_{2n+1}-P_{2n+1}^{-1}}\right)$

$$= \tan^{-1}\left(\frac{2\sqrt{2}}{Q_{2n}}\right) + \tan^{-1}\left(\frac{2\sqrt{2}}{Q_{2n+2}}\right).$$

**Theorem 6:** $\sum_{n=1}^{\infty}\tan^{-1}\frac{2}{P_{2n+1}} = \tan^{-1}\frac{1}{2}$.

**Proof:** By equation (2),

$$\sum_{n=1}^{m}\tan^{-1}\frac{2}{P_{2n+1}} = \sum_{n=1}^{m}\left(\tan^{-1}\frac{1}{P_{2n}} - \tan^{-1}\frac{1}{P_{2n+2}}\right)$$

$$= \tan^{-1}\frac{1}{P_2} - \tan^{-1}\frac{1}{P_{2m+2}}$$

$$= \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{P_{2m+2}}$$

Hence,

$$\sum_{n=1}^{\infty}\tan^{-1}\frac{2}{P_{2n+1}} = \lim_{m\to\infty}\sum_{n=1}^{m}\tan^{-1}\frac{2}{P_{2n+1}}$$

$$= \lim_{m\to\infty}\left(\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{P_{2m+2}}\right)$$

$$= \tan^{-1}\frac{1}{2} - \lim_{m\to\infty}\tan^{-1}\frac{1}{P_{2m+2}}$$

**Theorem 7:** Let $p_n(x)$ denote the Pell Polynomial, then

$$\sum_{n=1}^{\infty} \tan^{-1}\frac{2x}{p_{2n+1}(x)} = \tan^{-1}\frac{1}{2x}.$$

**Proof:** Let $\theta_n = \dfrac{1}{p_n(x)}$ then

$$\tan\left(\theta_{2n} - \theta_{2n+2}\right) = \frac{\dfrac{1}{p_{2n}(x)} - \dfrac{1}{p_{2n+2}(x)}}{1+\dfrac{1}{p_{2n}(x)}\times\dfrac{1}{p_{2n+2}(x)}}$$

$$= \frac{p_{2n+2}(x) - p_{2n}(x)}{p_{2n+2}(x)p_{2n}(x)+1}$$

$$= \frac{2xp_{2n+1}(x)}{p_{2n+1}^2(x)}$$

$$\left(\begin{array}{l} \because \quad p_{n+2}(x) = 2xp_{n+2}(x) + p_n(x) \\ and \;\; p_{k-1}(x)p_{k+1}(x) - p_k^2(x) = (-1)^k \end{array}\right)$$

$$= \frac{2x}{p_{2n+1}(x)}$$

$$\theta_{2n} - \theta_{2n+2} = \tan^{-1}\frac{2x}{p_{2n+1}(x)}$$

Hence,

$$\tan^{-1}\frac{1}{p_{2n}(x)} - \tan^{-1}\frac{1}{P_{2n+2}(x)} = \tan^{-1}\frac{2x}{P_{2n+1}(x)}$$

$$\sum_{n=1}^{m}\tan^{-1}\frac{2x}{P_{2n+1}(x)} = \sum_{n=1}^{m}\tan^{-1}\frac{1}{p_{2n}(x)} - \tan^{-1}\frac{1}{P_{2n+2}(x)}$$

$$= \tan^{-1}\frac{1}{p_2(x)} - \tan^{-1}\frac{1}{P_{2m+2}(x)}$$

$$= \tan^{-1}\frac{1}{2} - \tan^{-1}0 = \tan^{-1}\frac{1}{2}.$$

Since $p_2(x) = 2x$ and $\tan^{-1}\dfrac{1}{P_{2m+2}(x)} \to 0$ as $m \to \infty$, this gives

$$\sum_{n=1}^{\infty}\tan^{-1}\frac{2x}{p_{2n+1}(x)} = \tan^{-1}\frac{1}{2x}.$$

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