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A Statistical Approach to Families of Triple Close Approaches Occuring in Stellar Systems in Three Dimensions (II)

Mathematics

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ABSTRACT

This is the second part of a series of papers dealing with the statistical approach to families of triple close approaches in three dimensional space. The three-dimensional case is completely different from the two dimensional case. In this paper, the complete statistical solutions (i.e. the distributions of eccentricity e of the binary, binary energy E_b *and the escape velocity of escaper* v_s *etc.) of the system are calculated and are in good agreement with the numerical results of dynamical evolution in the range of* $10^{-1} \le v_0 \le 10^{-10}$ *and direction of* $v_0 (0 \le \alpha, \beta \le \pi)$. *We have also applied double limit process as in two dimensional study of our previous paper (I) to the system in three dimensional space and observed that the perturbing velocity* $v_0 \to 0^+$, *the product of the semi-major axis 'a' of the final binary and the square of the escape velocity* v_∞ *i.e.* $a v_\infty^2 \to 2/3$, *whatever direction of* v_0 *may be.*

1. Introduction:

This paper deals triple close approaches in three dimensional space with statistical approach. We draw attention to areas of astrophysics where the three-body problem has been obtained a central position and also discuss some new areas where its importance has still to be proven. In this field, the dynamical evolution of triple encounters in the stellar system has attracted the attention of researchers for a long time. Here, we wish to move to the three-body solution when the trajectories depend sensitively on the initial conditions.

The conjecture of Birkhoff [5,6] and latter reformulated by Szebehely [16,17] known as Birkhoff-Szebehely conjecture states that sufficiently triple close simultaneously asymmetric approach results with the formation of a binary and a escape of the third body. It seems to be of fundamental importance in the global behavior of three gravitational interacting stars. The dynamical evolution of such system has been studied by many authors Agekian and Anogova [1] Szebehely [16,17,18], Ansova and Orlov [3], Valtonen [20], Valtonen and Mikkola [23], Chandra and Bhatnagar [7] and others.

The study of triple systems in comparison with a two-body systems is more useful, but it is far more difficult to study it analytically as well as numerically. The reason is that a two-body systems with negative total energy are always bounded while triple systems are not. It is convenient to study a three-body system by replacing it with two two-body systems. In this process, the original three-body system approaches its partition into two two-body systems, in which the escaper and the binary form a two two-body system and the members of the binary form another two-body system. In fact, the main problem is the partition of the phase space of the initial conditions. The region of phase space with bounded motion is mixed with escape regions according to Henon [9].

Since Sundman [15] has shown that simultaneously close approaches occur only with small values of the total angular momentum **C** (in fact for triple collision **C** = 0 is a necessary condition) and study of system with low values of angular momentum **C** favour escape. The dynamical evolution of rotating triple system has been numerically studied by many authors (Anosova [1], Standish [14], Saslaw, Valtonen and Aarseth [24], Ansova, Bentov and Orlov [2], Mikkola and Valtonen [10], Ansova and Orlov [4] etc.) and showed that the angular momentum is an important parameter in the description of the final state.

In addition to numerical dynamical evolutions, the dynamical disruption of triple system has been studied with statistical theories by so many authors; Heggie [8], Monaghan [11,12], Nash and Monaghan [13], Valtonen et al.[19] etc.

Both methods have their advantage and disadvantage. Here, the subject is the general problem of three-body of the equilateral Lagrangian solution in symmetric rotating configuration. The masses of the participating bodies are taken equal. In Lagrangian solution, the symmetric configuration is never destroyed, therefore, escape does not occur and all motion are periodic, even when angular momentum C is small. If C is small and asymmetric changes of the initial conditions are introduced, it leads to escape instead of periodic orbits. The existence of such type of behavior are along the previously mentioned Birkhoff-Szebehely conjecture. For this, the triple close approach problem with three-parameter families has been studied by Chandra and Bhatnagar [7] in which, they have shown that how a symmetric triple collision is avoided by imparting very small velocity v_0 in a direction of $0 \le \alpha, \beta \le \pi$. Our aim here is to study the problem analytical and precise numerical investigation in the framework of statistical escape theories which is developed by Monaghan [11,12] and Valtonen etl.[19]. This statistical theories of the disruption of triple system are based on the assumption that the phase trajectory of triple system is quasi-ergodic within the region of close triple approach. Then the probability of escape with certain orbital elements of the final binary and the escaping body is proportional to the corresponding volume of phase space in a coordinate system which is associated with the centre of mass of the triple system. According to Szebehely [25] and Agekian et al. [26] classification, the families presented in this paper belongs to the classification 'O'. Furthermore, our aim is also to study the problem with double limit-process.

2.Statement of the Problem

Three equal masses are considered to occupy initially the vertices of an equilateral triangle $P_1P_2P_3$. These particles in the absence of any disturbance will move along

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the medians of the triangle and collide at the centroid of the triangle. Here, to avoid such a collision the mass at P_3 is subjected to small perturbing velocity v_0 in any direction in space. The masses at P_1 and P_2 are also subjected to perturbing velocity $v_0/2$ parallel and opposite to v_0. The centre of mass of the system stays at the centroid of the triangle for all times. We wish to study with statistical approach for what values of v_0 and direction of the participating bodies, the asymmetry will result in the formation of a binary and escape of the third body in three dimensional space.

Let there be three equal masses $m_1 = m_2 = m_3$ each equal to unity, occupy the vertices of an equilateral triangle $P_1P_2P_3$, the centre of mass 'O' of the system as the origin and x-axis parallel to the side of the triangle P_1P_2 and z-axis perpendicular to the plane $P_1P_2P_3$. We also choose the unit of distance such that at t = 0, $P_1P_2 = P_2P_3 = P_3P_1 = 1$ (see figure-1, Chandra and Bhatnagar [7]).

By symmetry each particle will be at the same distance r from the origin. The masses at P_1, P_2, P_3 are subjected to small perturbing velocity $-\mathbf{v}_0/2$, $-\mathbf{v}_0/2$, $-\mathbf{v}_0$ respectively such that $\mathbf{v}_0 = v_0\left[\text{Cos}\,\alpha\,\mathbf{i} + +\text{Cos}\,\beta\,\mathbf{j} + \text{Cos}\,\gamma\,\mathbf{k}\right]$. At t = 0, the position (x_i, y_i, z_i) and velocities $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$, i = 1,2,3 of P_1, P_2, P_3 are given by $x_1 = 1/2, x_2 = -1/2, x_3 = 0$,

$y_1 = y_2 = -1/2\sqrt{3}, y_3 = 1/\sqrt{3}$,

$z_1 = z_2 = z_3 = 0$, $\quad \dot{x}_1 = \dot{x}_2 = -v_0\text{Cos}\alpha/2$;

$\dot{x}_3 = v_0\text{Cos}\alpha$, $\dot{y}_1 = \dot{y}_2 = -v_0\text{Cos}\beta/2$,,

$\dot{y}_3 = v_0\text{Cos}\beta$, $\quad \dot{z}_1 = \dot{z}_2 = -v_0\text{Cos}\gamma/2$,

$\dot{z}_3 = v_0\text{Cos}\gamma$.

The introduction of perturbing velocities break the symmetry and collision can be avoided. The distances between the masses are functions of $v_0, \alpha, \beta, \gamma$, and t.

Let r_{12}, r_{23}, r_{31} represent the lengths of the sides P_1P_2, P_2P_3, P_3P_1 respectively. These three distances are not equal any more. In subsequent motion the values of distances up to the second order of time t are expressed as

$$r_{12} = 1 - \frac{3}{2}t^2;$$

$$r_{23} = r_{12} + \frac{3}{4}v_0\left(\text{Cos}\alpha + \sqrt{3}\text{Cos}\beta\right)t + \frac{9}{8}v_0^2\left(1 - \frac{1}{4}(\text{Cos}\alpha + \sqrt{3}\text{Cos}\beta)^2\right)t^2;$$

$$r_{31} = r_{12} - \frac{3}{4}v_0\left(\text{Cos}\alpha - \sqrt{3}\text{Cos}\beta\right)t + \frac{9}{8}v_0^2\left(1 - \frac{1}{4}(\text{Cos}\alpha - \sqrt{3}\text{Cos}\beta)^2\right)t^2.$$

It may be observed that r_{12} is independent of v_0 and α while r_{23} and r_{31} are functions of both to the order of t^2.

For the present study we require some more parameters at $t = 0$, which are given below:

(i) Moment of Inertia I

$$I(t) = \sum_{i=1}^{3} m_i r_i^2 = 1 + \sqrt{3}\,v_0\,\text{Cos}\beta\;t + \frac{3}{2}\left(v_0^2 - 2\right)t^2 + O(t^3),$$

,

At $t = 0$, $I(0) = 1$

$\dot{I}(0) = \sqrt{3}\,v_0\,\text{Cos}\beta\;t, \ddot{I}(0) = 3v_0^2 - 6.$

(ii) Total Energy E_t

$$E_t = K_t + V_t = \sum_{i=1}^{3}\frac{1}{2}(m_i\dot{\mathbf{r}}_i^2) - G\sum_{1\le i<j\le 3}\frac{m_i m_j}{r_{ij}}.$$

At $t=0$, $E_t = 3\left(\frac{v_0^2}{4}-1\right)$, where K_t and V_t are kinetic and potential energy of the system.

(iii) **Angular Momentum**

$\mathbf{C}^2 = \left[\sum_{i=1}^{3} m\mathbf{r}_i \times \dot{\mathbf{r}}_i\right]^2$. At t = 0, $|\mathbf{C}| = \frac{\sqrt{3}}{2}v_0 \,\mathrm{Sin}\beta$ and having direction cosines

$$\left(\frac{\mathrm{Cos}\gamma}{\mathrm{Sin}\beta}, 0, \frac{\text{-}\mathrm{Cos}\alpha}{\mathrm{Sin}\beta}\right).$$

3.Possibility of Escape

Since the distance of the escaper from the two-body which form a binary will go on increasing and eventually will be greater than the relative distance between the binary, the body which is likely to escape must be opposite to the shortest distance between the participating bodies. Keeping this criteria in view, it may be observed from the relative distances r_{12}, r_{23}, r_{31} of the participating bodies that when low value of $\mathbf{v_0}$ is introduced the initial symmetry gets destroyed and gives rise to escape of one of the bodies which is opposite to the smallest side provided escape conditions are satisfied.

In order to see which body is likely to escape and the other two forming a binary for low values of $\mathbf{v_0}$ = $v_0(0 \le \alpha, \beta \le \pi)$ are analysed on the basis of relative distances of the participating bodies. Our result agrees with Chandra and Bhatnagar [7]. The possibility of escape with the formation of binary in three-dimensional space are given in table-I.

Table I

Possible region of escape with the formation of a binary

Case↓	Inequality of distances	Possibility of escape	Binary
(I) $\mathrm{Cos}\alpha > 0, \mathrm{Cos}\beta > 0$;			
(a) $\mathrm{Cos}\alpha - \sqrt{3}\mathrm{Cos}\beta > 0$,	$r_{12} < r_{31} \le r_{23}$	m_2	m_3, m_1
(b) $\mathrm{Cos}\alpha - \sqrt{3}\mathrm{Cos}\beta \le 0$,	$r_{12} < r_{31} \le r_{23}$	m_3	m_1, m_2
(II) $\mathrm{Cos}\alpha < 0, \mathrm{Cos}\beta > 0$;			
(a) $\mathrm{Cos}\alpha + \sqrt{3}\mathrm{Cos}\beta \ge 0$,	$r_{12} < r_{23} < r_{31}$	m_3	m_1, m_2
(b) $\mathrm{Cos}\alpha + \sqrt{3}\mathrm{Cos}\beta < 0$,	$r_{23} < r_{12} < r_{31}$	m_1	m_2, m_3
(III) $\mathrm{Cos}\alpha > 0, \mathrm{Cos}\beta < 0$;			
(a) $\mathrm{Cos}\alpha + \sqrt{3}\mathrm{Cos}\beta \ge 0$,	$r_{31} < r_{12} \le r_{23}$	m_2	m_3, m_1
(b) $\mathrm{Cos}\alpha + \sqrt{3}\mathrm{Cos}\beta < 0$	$r_{31} < r_{23} < r_{12}$	m_2	m_3, m_1
(IV) $\mathrm{Cos}\alpha < 0, \mathrm{Cos}\beta < 0$;			
(a) $\mathrm{Cos}\alpha - \sqrt{3}\mathrm{Cos}\beta \le 0$,	$r_{23} < r_{12} \le r_{31}$	m_1	m_2, m_3
(b) $\mathrm{Cos}\alpha + \sqrt{3}\mathrm{Cos}\beta > 0$	$r_{23} < r_{31} < r_{12}$	m_1	m_2, m_3
(V) $\mathrm{Cos}\alpha = 0, \mathrm{Cos}\beta \ge 0$, or $\mathrm{Cos}\alpha = 0, \mathrm{Cos}\beta < 0$,	Decision can not be taken		
(VI) $\mathrm{Cos}\alpha > 0, \mathrm{Cos}\beta = 0$;	$r_{31} < r_{12} \le r_{23}$	m_2	m_3, m_1
(VI) $\mathrm{Cos}\alpha < 0, \mathrm{Cos}\beta = 0$;	$r_{23} < r_{12} \le r_{31}$	m_1	m_2, m_3

4. Analytical Theory

The three-body system approaches its partition into two two-body systems, in which the escaper and the centre of mass of the binary form a hyperbolic two-body system and the member of the binary form another an elliptic two-body system.

After the binary was formed, the total energy of the system is E_t may be written as

$$E_t = E_s + E_b \ , \qquad (1)$$

where, E_s is the escape energy, E_b is the stored energy in the binary. Equations for E_s and E_b may be written from two-body consideration as follows:

$$E_s = \frac{1}{2} M_2 \dot{\mathbf{r}}_s^2 - G \frac{m_s m_b}{r_s},$$

$$E_b = \frac{1}{2} M_1 \dot{\mathbf{r}}^2 - G \frac{m_a m_b}{r},$$

where m_s is mass of third body (escaper) and m_a and m_b are masses of the binary components. The position vector of third body (escaper) relative to the barycentre of the binary is $\mathbf{r_s}$ while the binary components are separated by the vector $\mathbf{r}_{ij} = \mathbf{r}$ $(1 \le i < j \le 3)$. We call total mass of the binary components $m_B = m_a + m_b$ and total mass of the system $M = m_B + m_s$. Then the reduced masses $M_1 = \frac{m_a m_b}{m_B}$ and $M_2 = \frac{m_B m_s}{M}$.

Several escape conditions exist in the literature. For instance, Standish [14] has shown that it is sufficient for escape that

$$E_s \ge \frac{G m_1 m_2 m_3}{m_1 + m_3} \frac{d^2}{r_s^2 (r_s - d)} + \frac{M m_2}{2(m_1 + m_3)} v_s^2 \operatorname{Sin}\mu,$$

and that $r_s > d = \frac{G}{|E_t|} \sum_{1 \le i < j \le 3} m_i m_j,$

where μ is the angle between $\mathbf{v}_s$ and $\mathbf{r}_s$.

These conditions are satisfied and escape does occur for sufficiently small values of $\mathbf{v_0}$ in a space. This follows from the fact that as $v_0 \to 0^+, I_m \to 0^+$, where, I_m = minimum moment of inertia. The asymmetric triple close approach, after reaching minimum moment of inertia I_m generates sufficiently large values of I and $\dot{I}$ for escape. For an indication of this process see Birkhoff [5] and Szebehely [17].

According to the above escape condition, m_1, m_3 form a binary and m_2 escapes. It has to be modified according to the binary components and escaper (Table –I).

In our case, the total mass of the system M = 3, the total mass of the binary $m_B = 2$ and the reduced masses $M_1 = 1/2$ and $M_2 = 2/3$

(I) Statistical Process

(A) Phase Space Volume

In the statistical theory of the disintegration of three-body systems by Monaghan's [11,12] and Valtonen [21], it is assumed that the probability of a given escape configuration is proportional to the volume in phase space available to this configuration. This assumption is supported by results from other theoretical approaches as well as from numerical orbit calculations.

The density σ of states (escape configuration in the phase space per unit energy) is obtained by integrating δ-function over the phase space volume with the phase space co-ordinates $\mathbf{r}, \mathbf{r}_s, \mathbf{p}, \mathbf{p}_s$ where $\mathbf{p}$ and $\mathbf{p}_s$ are the canonical momenta associated with $\mathbf{r}$ and $\mathbf{r}_s$ respectively.

$$\sigma = \iint\iint \delta\left(\frac{P_s^2}{2M_2} + V_s + E_b - E_t\right) d\mathbf{r}_s, d\mathbf{p}_s, d\mathbf{r}, d\mathbf{p} \tag{2}$$

Where we have put $E_s = \frac{P_s^2}{2M_2} + V_s$, $\left(\mathbf{P}_s = \frac{2}{3}\dot{\mathbf{r}}_s\right)$.

To determine the properties of the binary, we integrate eqn. (2) over $\mathbf{r}_s$ and $\mathbf{p}_s$ and for escaper we integrate over $\mathbf{r}$ and $\mathbf{p}$. Firstly we carry out the integrations over the momentum space $\mathbf{p}_s$ with a uniform distribution of directions over the whole sphere. From the rules for integrating δ-functions

$$\sigma = \int \delta\left(\frac{P_s^2}{2M_2} + V_s + E_b - E_t\right) d\mathbf{p}_s$$

$$= 4\pi \int_0^\infty \delta\left(\frac{P_s^2}{2M_2} + V_s + E_b - E_t\right) P_s^2 \, dP_s$$

$$= 4\pi M_2 \int_0^\infty \delta\left[x - (E_t - V_s - E_b)\right] \sqrt{2M_2 x}\, dx$$

$$= 4\pi M_2 \sqrt{2M_2(E_t - V_s - E_b)} \; . \tag{3}$$

Where $x = P_s^2 / 2M_2$. From the property of the δ- function: $\int_0^\infty f(x)\delta(x - a)dx = f(a)$ = 1, where $f(x)$ is any function.

Monaghan [11] integrates over $\mathbf{r}_s$ and $\mathbf{p}_s$, assuming that the escaper orbit is radially outword from the barycentre of the binary while Valtonen [21] assume that the escaper at a distance $\mathbf{r}_s$ from the binary centre has come to this point along a straight line orbit. It must have come from the neighbourhood of the binary. Otherwise it would not have acquired the escape velocity. The neighbourhood of the binary may be defined for our purposes as a circular area, perpendicular to the vector over $\mathbf{r}_s$, with the radius of some simple multiple n a, where a is the semi-major axis of the binary and the multiple n comes from experiences with orbit calculations and is $\approx$ 7. The semi - major axis 'a' of the binary is related to the binary energy by

$$a = -\frac{m_a m_b}{2E_b} \tag{4}$$

The straight –line drawn from the point $\mathbf{r}_s$ through the circle of radius 7a define a cone; this is called the "loss cone" because particle travelling in reverse direction from the apex of the cone to the binary will generally be scattered away from the cone and thus these orbits are lost. In the current problem we can say that only the orbits within the loss cone are true escape orbits. Since they have been strongly influenced by the binary in the past. The loss cone direction contain approximately the fraction of $\pi(7a)^2 / 4\pi r_s^2 = 12.25(a/r_s)^2$ of the whole sphere of radius r_s surrounding the escaper. Since δ- functions does not depend on $\mathbf{r}_s$. We can immediatlly perform the integration taking account of the loss cone factor $(49/4)(a/r_s)^2$ with $d\mathbf{r}_s = 4\pi r_s^2 \, dr_s$ and second integral becomes

$$4\text{x}49\pi^2\sqrt{2}M_2^{3/2}\text{a}^2\int_0^R\sqrt{E_t-E_b+(Gm_sm_B/r_s)}\ \text{dr}_s$$

. (5)

Where the upper limit R of the $\mathbf{r}_s$ range is considered a free parameter. Let us denote

$$x^2=r_s, i.e. 2x\,dx=dr_s \text{ and } y=\frac{(E_t-E_B)}{(Gm_sm_B/R)}$$

Then integration is easily carried out and we get

$$\int_0^{\sqrt{R}} 2\sqrt{Gm_sm_B}\,\sqrt{(yx^2/R)+1}\ \text{dx}$$

$$=\sqrt{Gm_sm_BR}\left[\sqrt{y+1}+\frac{1}{\sqrt{y}}\log(\sqrt{y}+\sqrt{y+1})\right].$$

The function in the square bracket has the value of 2.3 when y = 1 and it approaches 2 when $y\to 0$. If R is relatively small, say $R=3a_0$, this is more or less the range of interest of y. As a slowly varying function of y the square bracket may therefore be put equal to 2. Here a_0 is the initial semi-major axis of the binary before the three-body interaction. Using $m_sm_B=M_2M$ and $a=-(Gm_am_b)/(2E_b)$ in equation (5), we get

$$\sigma=98\sqrt{2}\,\pi^2(\text{GMR})^{1/2}\text{M}_2(\text{Gm}_a\text{m}_b)^2.$$
$$\int\ldots\int(\text{d}\mathbf{r}\,\text{d}\mathbf{p})/|\text{E}_b|^2$$

(6)

In order to see the significance of the remaining integrals, we use polar co-ordinates (r,θ,ϕ). Then $\theta\to\pi/2-\theta$ and now we measure the $\theta-$coordinate from the pole rather than from the equator. We write the remaining integral in the form

$$\int\text{d}\mathbf{r}\,\text{d}\mathbf{p}=\iint\iint\text{dr}\,d\theta\,d\phi\,\text{dp}_r\,\text{dp}_\theta\,. \qquad (7)$$

With the change of variable,

$$\text{E}_b=\frac{1}{2}\frac{\text{P}^2}{\text{M}_1}-\frac{\text{Gm}_a\text{m}_b}{\text{r}}$$

$$=\frac{1}{2\text{M}_1}\left(\text{p}_r{}^2+(\text{P}_\theta{}^2/r^2)+(P_\phi{}^2/r^2\sin^2\theta)\right)-$$
$$(Gm_am_b/r),$$

from which

$dE_b/dP_r=P_r/M_1, i.e. dP_r=(M_1\,dE_b)/P_r$ and

$$P_r=\begin{bmatrix}2M_1E_b+(2M_1Gm_am_b/x)-(k^2/x^2)-\\(P_\phi{}^2/x^2\sin^2\theta)\end{bmatrix}^{1/2}$$

, where $x\equiv r$ and $k\equiv P_\theta$.

The square of the total angular momentum vector is

$L^2=k^2+(P_\phi{}^2/\sin^2\theta)$, then

$P_\phi=\sin\theta\sqrt{L^2-k^2}$ and

$dP_\phi=(\sin\theta\,L\,dL)/\sqrt{L^2-k^2}$,

from which we get

$$P_r=\left[2\text{M}_1\text{E}_b+(2M_1\text{Gm}_a\text{m}_b/x)-(L/x)^2\right]^{1/2}$$

The integral in eqn.(7) becomes

$$\iint\frac{dE_b}{E_b{}^2}\frac{\text{M}_1\,dk\,(\sin\theta/\sqrt{L^2-k^2})\,\text{L}\,\text{dL}\text{d}x\,\text{d}\theta\,d\phi}{\left[2M_1\text{E}_b-(\text{L}/x)^2+(2\text{Gm}_a\text{m}_b\text{M}_1/x)\right]^{1/2}}$$

(8)

The integral over θ and ϕ give $\int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\phi = 4\pi$.

The integral of k is

$$\int_{-L}^{L} dk \Big/ \sqrt{L^2 - k^2} = \Big|_{-L}^{L} arc\sin(k/L) = \pi \,.$$

The integral over x is

$$\int_{x_2}^{x_1} \frac{x\,\mathrm{d}x}{\left[-\mathrm{L}^2 + 2\mathrm{Gm_a m_b} M_1 x - 2\mathrm{M}_1 \,|\,\mathrm{E_b}\,|\, x^2\right]^{1/2}}$$

After evaluation of total contribution of the integration over x becomes

$$\frac{\pi}{2\sqrt{2}} \frac{\mathrm{G(m_a m_b)}^{1/2} \mathrm{m_B}^{1/2}}{\left|\mathrm{E_b}\right|^{3/2}} .$$

To integrate L it is more useful to transform to the eccentricity e with the binary orbit formula

$$\mathrm{L}^2 = \mathrm{M}_1 \frac{(\mathrm{Gm_a m_b})^2}{2\left|\mathrm{E_b}\right|}(1 - \mathrm{e}^2), \text{and}$$

$$\mathrm{L\,dL} = \mathrm{M}_1 \frac{(\mathrm{Gm_a m_b})^2}{2\left|\mathrm{E_b}\right|} \mathrm{e\,de} .$$

Combining the integrations over θ, ϕ, k and x together with eqn. (5), we find the density of states becomes

$$\sigma = 98\pi^5 M_1 M^{-3/2} \mathrm{R}^{1/2} m_{\mathrm{B}}^{3/2} m_s^{2} (\mathrm{Gm_a m_b})^{11/2} . \iint \frac{\mathrm{dE_b}}{\left|\mathrm{E_b}\right|^{9/2}} e\, de. \quad (9)$$

It may be noted here that we consider only the quantities which follow the integral signs and leave the coefficients in front of them. These quantities represent the distribution over which one has to integrate in order to obtain the total phase space volume. These are the fundamental distribution in which we are interested. Thus the distribution of binary energy |Eb|, normalized to unity, is

$$f(\left|\mathrm{E_b}\right|)\, d\left|\mathrm{E_b}\right| = 3.5\left|\mathrm{E_t}\right|^{7/2} \left|\mathrm{E_b}\right|^{-9/2} d\left|\mathrm{E_b}\right|. \quad (10)$$

This result may also be derived from other theoretical concept like Heggie [8] and from such concept, equation (10) represents an equilibrium distribution.

The corresponding distribution of eccentricity is

$$f(\mathrm{e}) = 2\mathrm{e\,de} . \quad (11)$$

The statistical theory therefore predicts that there is no correlation between the binary energy $\mathrm{E_b}$ and eccentricity e of the binary. Since the semi-major axis a is proportional to $1/\mathrm{E_b}$ an equivalent prediction is that a and e are independently distributed.

(B) Escape Velocity:

When the binary energy $\mathrm{E_b}$ is known, then escape energy $\mathrm{E_s} = \left|\mathrm{E_b}\right| - \left|\mathrm{E_t}\right|$ is obtained from the equation (1). We substitute this in equation (9) and integrate over e, i.e.

$\int_0^1 \mathrm{e\,de} = 1/2$, after which there remains

$$\sigma = 49\pi^5 m_s^{2} m_{\mathrm{B}}^{3/2} M_1 M^{-3/2} R^{1/2} (\mathrm{Gm_a m_b})^{11/2} . \iint \frac{\mathrm{dE_s}}{\left(|\,\mathrm{E_t}\,| + |\,\mathrm{E_s}\,|\right)^{9/2}} .$$

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We assume that the velocity of the escaper in the centre of mass coordinate system is v_s. Then $v_s = \frac{m_B}{M}|\dot{\mathbf{r}}_s|$, and

$$E_s = \frac{1}{2}M_2|\dot{\mathbf{r}}_s|^2 = \frac{1}{2}\frac{m_s M}{m_B}{v_s}^2 \qquad (12)$$

Therefore

$dE_s = \frac{m_s M}{m_B} v_s dv_s$, and

$$\sigma = 49\pi^5 M_1 M^{-1/2} R^{1/2} {m_s}^3 {m_B}^{1/2} (Gm_a m_b)^{1/2}. \int v_s dv_s / \left(|E_t| + \frac{1}{2}(m_s M/m_B){v_s}^2\right)^{9/2} \qquad (13)$$

From the above, the escape velocity can be written as $v_s = \left(\frac{2m_B E_s}{m_s M}\right)^{1/2}$. In our case, this escape velocity becomes $v_s = \frac{2\sqrt{E_s}}{\sqrt{3}}$, which depends on the escape energy.

$$f(v_s)\,dv_s = \frac{(3.5|E_t|^{7/2}\, m_s M/m_B) v_s dv_s}{\left(|E_t| + \frac{1}{2}\left(m_s M/m_B){v_s}^2\right)\right)^{9/2}} \qquad (14)$$

and the peak of this distribution is obtained

$$(v_s)_{peak} = \frac{1}{2}\sqrt{\frac{(M - m_s)}{m_s M}}\sqrt{|E_t|} \qquad (15)$$

when putting $df(v_s)/dv_s = 0$.

In our case, the peak of the escape velocity distribution is

After normalization, the escape velocity distribution becomes

The asymmetric triple close approach results in an escaper with the formation of a binary. The (asymptotic) hyperbolic escape velocity relative to the centre of mass of the system, v_s and the asymptotic value of the semi-major axis of the binary 'a' depend on the perturbation measured by $\mathbf{v_0}$. As $v_0 \to 0^+$, whatever direction may be, the product $a\,{v_s}^2 \to \frac{2}{3}$.

To establish such concept, first of all, attention is directed to the double limit process involved in the above result. In this limit process, the original three-body problem approaches its partition into two two-body problems. The escaper and the centre of mass of the binary form a hyperbolic two-body problem and the members of the binary form an elliptic two-body problem. After the binary is formed, the distance between the escaper and the binary, $r_s \to \infty$, the velocity of the escaper $v_s \to v_\infty$.

$$(v_s)_{peak} = \frac{1}{\sqrt{2}} = 0.707106781..$$

(I I) Double Limit Process:

The second limit process refers to the behavior of the members of the three-parameter families as the perturbing initial velocity $v_0 \to 0^+$, whatever direction may be. Thus the above result may be written as $a\,{v_\infty}^2 \to \frac{2}{3}$.

We have total energy of the system, $E_t = E_b + E_s$.

In the first limit process E_t is fixed,

$$E_b \to -G\frac{m_a m_b}{2a}, \text{ and } E_s \to \frac{1}{2}\frac{m_s M}{m_B} v_s^{\ 2},$$

Therefore,

$$E_t = \frac{1}{2}\frac{m_s M}{m_B} v_\infty^{\ 2} - \frac{G m_a m_b}{2a}.$$

In the second limit process, we consider the general form of the total energy

$$E_t = \frac{3}{4} m v_0^{\ 2} - \frac{3Gm^2}{l},$$

where $m = m_1 = m_2 = m_3$ and l is the length of the side of the equilateral triangle at the beginning of the motion.

Equating these two forms of the above total energy, we have

$$a\, v_\infty^{\ 2} = G\frac{m_a m_b}{m_s}\left(\frac{m_B}{M}\right) + \frac{3}{2} a \left(\frac{m m_B}{M m_s}\right).$$
$$\left(v_0^{\ 2} - 4\frac{Gm}{l}\right)$$

As $v_0 \to 0^+$, the following limiting values are obtained:

Minimum moment of inertia $I_m \to 0^+$, the angular momentum $\mathbf{C} \to 0^+$, the total energy $E_t \to -3Gm^2 l^{-1}$, the semi-major axis $a \to 0^+$ and the escape velocity $v_\infty \to \infty$, and consequently,

$a\, v_\infty^{\ 2} \to G\dfrac{m_a m_b}{m_s}\left(\dfrac{m_B}{M}\right)$. In our case,

$$a\, v_\infty^{\ 2} \to \frac{2}{3}.$$

It may be noted that if the above simplifying substitution is made prior to the second limit process, we have

$$a\left(v_\infty^{\ 2} - v_0^{\ 2} + 4\right) = \frac{2}{3}, \text{or}$$
$$v_\infty^{\ 2} = \left(v_0^{\ 2} + 2(3a)^{-1} - 4\right)$$

allowing an orderly presentation of numerically established members of the family.

It is essential to point out that the process allows the generation of arbitrary high-escape velocities with arbitrary close binaries.

5. Numerical Analysis

As the purpose of this paper is to investigate the results of triple close approaches with the formation of binary and the systematic regularity of escape of the third body in the framework of statistical escape theories (Valtonen and Myllari [19]). For this, we have studied the problem with small perturbing velocities $v_0 (10^{-1} \le v_0 \le 10^{-10})$ and their directions $(0^o \le \alpha, \beta \le \pi)$.

Chandra and Bhatnagar [7] has observed in actual experiment of dynamical evolution of the problem that two close approaches occur before the formation of a binary between the participating bodies. The first close approach means the first minimum smallest relative distance r_{ij} between the participating bodies. This occurs before minimum moment of inertia I_m is attained. And the body which actually escape is opposite to this smallest relative distance r_{ij}. The second close approach means the smallest relative distance between the participating bodies when I_m is attained. This minimum moment of inertia I_m is attained (second close approach) slightly latter than the first close approach. The two bodies amongst the participating bodies

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which has the first close approach are different from the two bodies which has the second close approach except for special regions. In special regions, bodies are the same and the first close approach occurs very slightly before attaining of I_m (second close approach).

Since triple close approaches results with the formation of a binary and escape of the third body when minimum moment of inertia I_m is attained. So, we have carried out experiment according to Chandra and Bhatnagar ([7], sec.3) to obtain the definite orbital elements of the final binary and the escaper when minimum moment of inertia I_m is attained for $10^{-1} \le v_0 \le 10^{-10}$, $0^o \le \alpha, \beta \le \pi$. For this, we have obtained semi-major axis 'a' from the maximum and minimum distances r_{ij} of the final binary. When semi-major axis 'a' is obtained from equation (4), then from equation (1), we have calculated binary energy E_b and escape energy E_s. We have also calculated eccentricity e, distribution of eccentricity $f(e)$, the magnitude of escape velocity v_s, distribution of escape velocity $f(v_s)$, distribution of $|E_b|$ i.e. $f(|E_b|)$. We have also adopted double limit process and calculated escape velocity which is denoted by v_∞ and it gives promising value as the value of escape velocity v_s (i.e. $v_s \cong v_\infty$).

It may be noted here that the total energy E_t of the system is -3 (i.e. $E_t < 0$), therefore $E_s = E_t - E_b = |E_b| - |E_t|$, and for escape $|E_b| > |E_t|$ is required. This is always established with sufficiently small value of semi-major axis 'a'. Therefore escape for negative total energy is associated with the formation of a close binary.

We define the following three families with the parameters v_0 and α, β, γ $(\text{Cos}^2\gamma = 1 - \text{Cos}^2\alpha - \text{Cos}^2\beta)$.

(i) For first family, varying α and keeping v_0 and β fixed,

(ii) For second family, varying β and keeping v_0 and α fixed,

(iii) For third family, varying v_0 and keeping α and β fixed.

In actual experiment, we have observed that how a condition of complete collapse may be perturbed to obtain well-established families of asymmetric triple close approaches with systematic regularity of escape with the formation of a binary which can be seen in table I. We have predicted this possibility of escape with the formation of a binary on the basis of relative distances of the participating bodies. As our main aim is to study in details the effect of the perturbing velocity $\mathbf{v_0} = v_0(\alpha, \beta, \gamma)$, on various parameters to obtain the complete statistical solutions (semi-major axis a, eccentricity e, binary energy E_b and its distribution $f(|E_b|)$, escape velocity of escaper v_s, and escape velocity distribution $f(v_s)$). So we have taken a typical representative family with $v_0 = 0.01$ having different directions α, β lying between 0 and π at an interval of 20^o (Table II).

In the columns of r_{ij}(max) and r_{ij}(min) of table II are maximum and minimum distances of the final binary against each angle α, β. These maximum and minimum distances are also known as Apastron and Periastron. Corresponding to these columns, the other columns give the values of semi-major axis a, binary energy E_b, escape

Table-II

Binary distance R_{ij}(max.) and R_{ij}(min.) , a, E_b, E_s, e, binary and escaper corresponding to α, β ($v_0 = 10^{-2}$).

Ang $\alpha\downarrow$	Ang. $\beta\downarrow$	Rij(max.)	Rij(min.)	Semi-maj axis a	Binary Energy E_b	Escape Energy E_s	Eccentricity e	Binary	Esca-per
0	90	1.880E-02	5.167E-03	0.0119835	-41.72403722	38.72403722	0.568824	m_2	m_1m_3
20	70	1.902E-02	7.823E-03	0.0134215	-37.253660	34.253660	0.417129	m_2	m_1m_3
	90	1.782E-02	4.900E-03	0.01136	-44.01408451	41.01408451	0.568662	m_2	m_1m_3
	110	2.276E-02	2.008E-03	0.012384	-40.3747	37.3747	0.837855	m_2	m_1m_3
40	50	4.139E-02	8.231E-03	0.0248105	-20.1527579	17.1527579	0.668245	m_3	m_1m_2
	70	2.746E-02	5.409E-03	0.0164345	-30.42380358	27.42380358	0.670875	m_2	m_1m_3
	90	1.490E-02	4.114E-03	0.009507	-52.59282634	49.59282634	0.567266	m_2	m_1m_3
	110	2.341E-02	2.639E-04	0.01183695	-42.24061097	39.24061097	0.977705	m_2	m_1m_3
	130	2.742E-02	2.173E-04	0.01381865	-36.18298459	33.18298459	0.984275	m_2	m_1m_3
60	30	2.099E-02	3.855E-03	0.0124225	-40.249547	37.249547	0.689676	m_3	m_1m_2
	50	1.406E-02	6.457E-03	0.0102585	-48.7400692	45.7400692	0.370571	m_3	m_1m_2
	70	1.111E-01	4.518E-04	0.0557759	-8.9644452	5.9644452	0.9919	m_3	m_1m_2
	90	1.019-02	2.874E-03	0.006532	-76.54623393	73.54623393	0.560012	m_2	m_1m_3
	110	1.788E-02	4.106E-05	0.00896053	-55.80027074	52.80027074	0.996418	m_2	m_1m_3
	130	2.027E-02	1.458E-03	0.010864	-46.02356406	43.02356406	0.865795	m_2	m_1m_3
	150	2.081E-02	3.825E-03	0.0123175	-40.59265273	37.59265273	0.689466	m_2	m_1m_3
80	10	2.732E-02	3.455E-04	0.01383275	-36.14610255	33.14610255	0.975023	m_3	m_1m_2
	30	2.446E-02	4.008E-04	0.0124304	-40.22396705	37.22396705	0.967756	m_3	m_1m_2
	50	1.869E-02	5.926E-04	0.0096413	-51.86022632	48.86022632	0.938535	m_3	m_1m_2
	70	9.988E-03	1.375E-03	0.0056815	-88.00492828	85.00492828	0.757986	m_3	m_1m_2
	90	3.745E-03	1.241E-03	0.002493	-200.562	197.562	0.552206	m_2	m_1m_3
	110	8.117E-03	2.319E-03	0.005218	-95.82215408	92.82215408	0.555577	m_2	m_1m_3
	130	1.248E-02	7.203E-03	0.0098415	-50.80526343	47.80526343	0.268099	m_2	m_1m_3
	150	2.663E-02	7.384E-03	0.017007	-29.39965896	26.39965896	0.565826	m_2	m_1m_3
	170	3.913E-02	7.723E-03	0.0234265	-21.34335048	18.34335048	0.670331	m_2	m_1m_3
100	10	2.738E-02	3.439E-04	0.01386195	-36.0699613	33.0699613	0.975191	m_3	m_1m_2
	30	2.444E-02	4.054E-04	0.0124227	-40.24889919	37.24889919	0.96736	m_3	m_1m_2
	50	1.865E-02	5.962E-04	0.0096231	-51.95830865	48.95830865	0.938045	m_3	m_1m_2
	70	9.989E-03	1.375E-03	0.005682	-87.99718409	84.99718409	0.758008	m_3	m_1m_2
	90	1.179E-02	3.346E-03	0.007568	-66.06765328	63.06765328	0.557875	m_1	m_2m_3
	110	8.103E-03	2.324E-03	0.0052135	-95.90486238	92.90486238	0.554238	m_1	m_2m_3
	130	1.248E-02	7.203E-03	0.0098415	-50.8053	47.8053	0.268099	m_1	m_2m_3
	150	2.663E-02	7.384E-03	0.017007	-29.3997	26.3997	0.565826	m_1	m_2m_3
	170	3.915E-02	7.741E-03	0.0234455	-21.3261	18.3261	0.66983	m_1	m_2m_3
120	30	2.101E-02	3.823E-03	0.0124165	-40.269	37.269	0.692103	m_3	m_2m_2
	50	1.406E-02	6.456E-03	0.010258	-48.7424	45.7424	0.370638	m_3	m_1m_2
	70	1.111E-01	4.501E-04	0.0557751	-8.96458	5.96458	0.99193	m_3	m_1m_2
	90	1.588E-02	4.434E-03	0.010157	-49.2271	46.2271	0.563454	m_1	m_2m_3
	110	1.786E-02	4.107E-05	0.00895054	-55.8626	52.8626	0.995411	m_1	m_2m_3
	130	2.026E-02	1.458E-03	0.010859	-46.0448	43.0448	0.865733	m_1	m_2m_3
	150	2.083E-02	3.812E-03	0.012321	-40.5811	37.5811	0.69061	m_1	m_2m_3
140	50	4.147E-02	8.231E-03	0.0248505	-20.1203	17.1203	0.668779	m_2	m_1m_3
	70	2.746E-02	5.409E-03	0.0164345	-30.4238	27.4238	0.670875	m_1	m_2m_3
	90	1.821E-02	5.059E-03	0.0116345	-42.9756	39.9756	0.565173	m_1	m_2m_3
	110	2.341E-02	2.639E-04	0.011837	-42.2406	39.2406	0.977705	m_1	m_2m_3

	130	3.181E-02	1.285E-06	0.0159056	-31.4354	28.4354	0.999919	m_1	m_2m_3
160	70	1.901E-02	7.843E-03	0.0134265	-37.2398	34.2398	0.415857	m_1	m_2m_3
	90	1.864E-02	5.147E-03	0.0118935	-42.0398	39.0398	0.567243	m_1	m_2m_3
	110	2.656E-02	6.489E-04	0.0136045	-36.7527	33.7527	0.952302	m_1	m_2m_3
180	90	1.888E-02	5.133E-03	0.0120065	-41.6441	38.6441	0.572482	m_1	m_2m_3

energy E_s, eccentricity e. The member of binary and escaping body are mentioned in last two columns.

In actual experiment, we have made observation from the above that for the semi-major axis a, eccentricity e, binary energy E_b and escape energy E_s fluctuate between the corresponding minimum and maximum values.

The following characteristics of the above mentioned families are further observed that

(I) The Distribution of Eccentricity $f(e)$ and e

The distribution of eccentricities in the breakup of planar three-body systems for all the family are prominent $(e < 1)$ in all respect. We have drawn the curves eccentricity distribution $f(e)$ vs e for all the families in figures 1a-1c. We have made observation that the curve fluctuate between the corresponding maximum and minimum values and all are less than 1.

Fig.1a. Eccentricity distributions f(e) vs. e

(α varies , $\beta = 80^o$, $v_0 = 10^{-2}$).

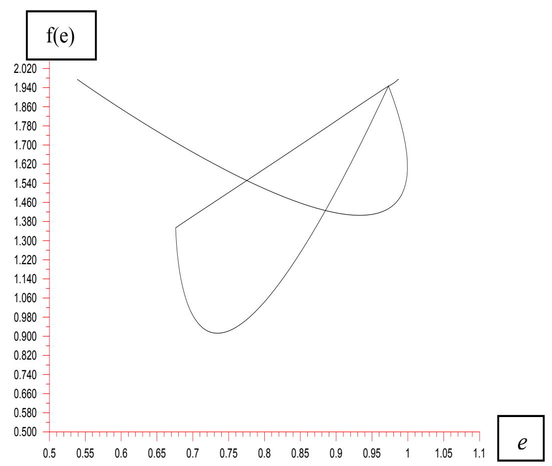

Fig.1b. Eccentricity distributions f(e) vs. e

(β varies , $\alpha = 80^o$, $v_0 = 10^{-2}$).

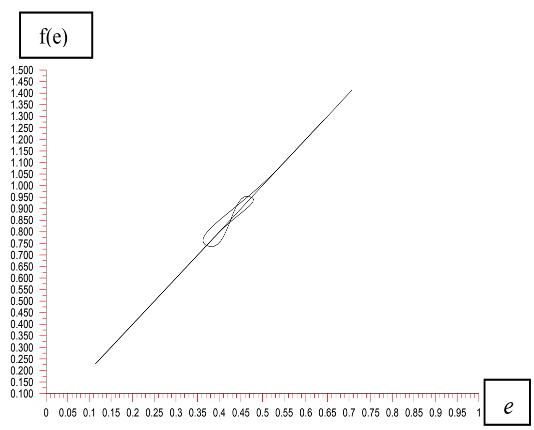

Fig.1c. Eccentricity distributions f(e) vs. e

(v_0 varies , $\alpha = 80^o$, $\beta = 80^o$).

(II) The Distribution of $|E_b|$ i.e. $f(|E_b|)$:

It is observed from fig.2a for the first family that the curve of $f(|E_b|)$ first decreases up to a certain value of α and then increases and get maximum pick point at $\alpha = 100^o$ thereafter curve decreases. For the second family, the curve decreases up to a certain value of α then increases (Fig.2b). And for the third family, the curve first decreases with v_0 increases up to certain value of α and then increases (Fig.2c).

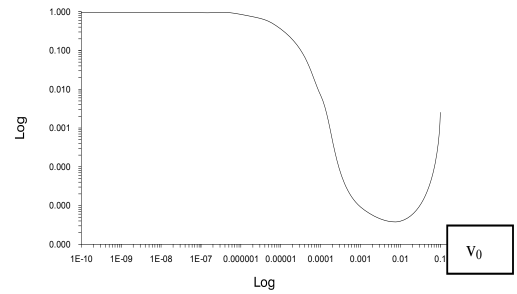

Fig.2c. Binary energy distributions $f(E_b)$ vs. v_0

($\alpha = 80^o$, $\beta = 80^o$).

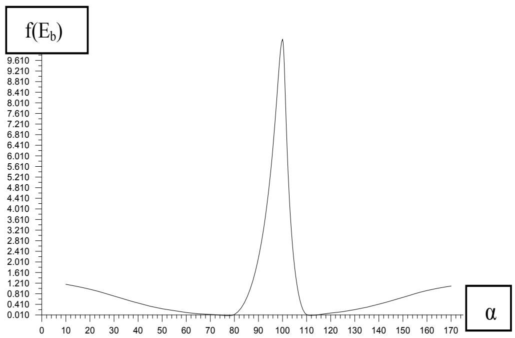

Fig.2a. Binary energy distributions $f(E_b)$ vs. α

($\beta = 80^o$, $v_0 = 10^{-2}$).

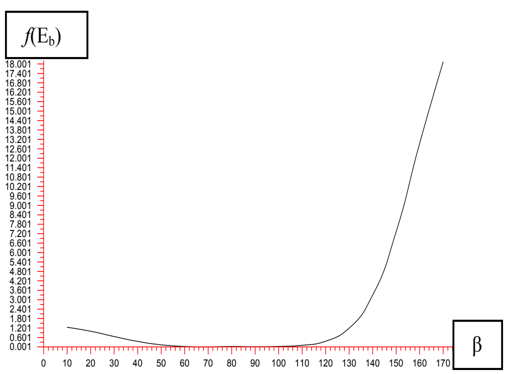

Fig.2b. Binary energy distributions $f(E_b)$ vs. β

($\alpha = 80^o$, $v_0 = 10^{-2}$).

$f(E_b)$

(III) The Escape Velocity v_s and its Distribution $f(v_s)$

It is observed for the first family that the escape velocity v_s first increases up to a certain value of α and then decreases whereas $f(v_s)$ first decreases up to a certain value of α and then increases (Fig.3a). For the second family trends of v_s and $f(v_s)$ are same as the first family (Fig.3b). And for the third family v_s increases with v_0 up to certain value and then rapid fall (Fig.3c).

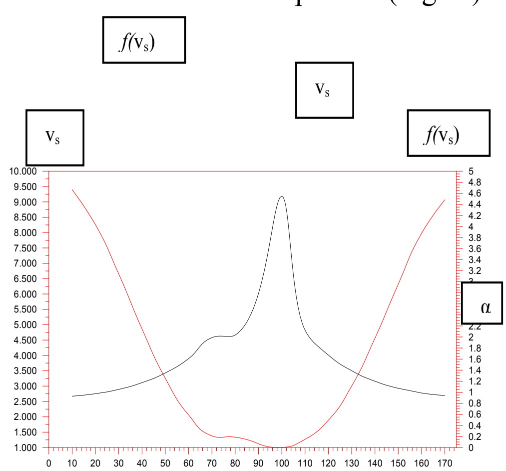

Fig.3a. Escape velocity v_s and escape velocity distributions $f(v_s)$ vs. α ($\beta = 80^o$, $v_0 = 10^{-2}$).

v_s $f(v_s)$

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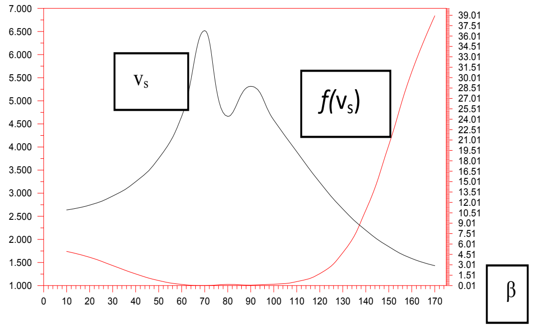
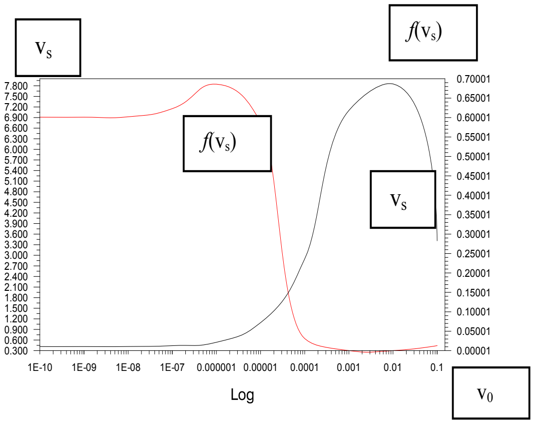


Fig.3b. Escape velocity v_s and escape velocity distributions

$f(v_s)$ vs. β (α = 80°, $v_0 = 10^{-2}$).



3c. Escape velocity v_s and escape velocity distributions

$f(v_s)$ vs. v_0 (α = 80°, β = 80°).

(IV) In Double Limit Process:

It is observed that the escape velocity v_∞ and v_s and v_s give promising value $(v_\infty \cong v_s)$ and $a\,v_\infty^{\;2} \to 2/3$ as $v_0 \to 0^+$, whatever direction of v_0 may be. We have shown the value of $A = a\,v_\infty^{\;2}$ for three families in figures 4a, 4b and 4c. It may be observed from these figures that the first family the value of A increase up to a certain value of α and then decreases. For the second family, the characteristic is same (Fig.4b.). And for third family, A increases with v_0 up to a certain value and then decreases.

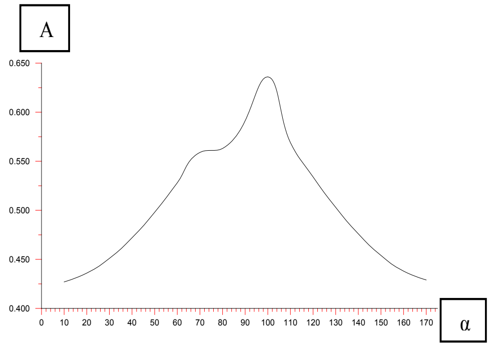


Fig.4a. $A(=a\,v_\infty^{\;2})$ versus. α, (β = 80°, $v_0 = 10^{-2}$).

It may be seen in fig.4c. However, In all three families A ultimately tends to 2/3. This support our above statement that $a\,v_\infty^{\;2} \to 2/3$, whatever the direction of $\mathbf{v_0}$ may be.

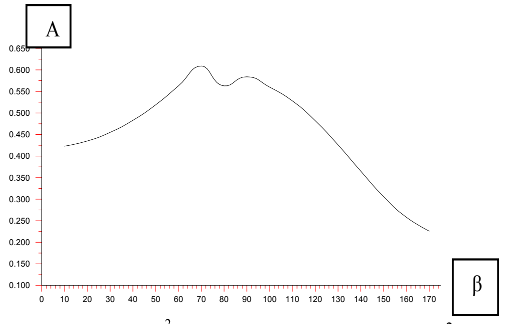


Fig.4b. $A(=a\,v_\infty^{\;2})$ versus. β, (α = 80°, $v_0 = 10^{-2}$).

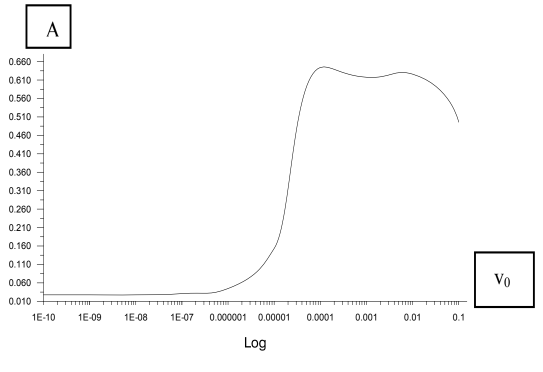


Fig.4c. . $A(=a\,v_\infty^{\;2})$ versus v_0 (α = 80°, β = 80°).

Conclusion

The dynamical escape with the formation of a binary from a triple system has been studied in the frame work of statistical escape theories in three dimensional space. We have used ergodic principle as a useful tool in the description of the statistical results of the three-body break-up in three dimensional space. This work is based upon the theory of Monaghan and later reformulated by Valtonen and associates. Its general principles are also confirmed in good agreement with numerical results of dynamical evolution as two dimensional problem. We have also demonstrated that the double limit process is also a useful tool in three dimensional space to describe the three-body break-up.

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