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Research Paper

A Statistical Approach to Two-Parameter Families of Triple Close Approaches in Stellar System (I)

Mathematics

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ABSTRACT

A statistical approach to two-parameter families of triple close approaches which result a systematic regularity of escape with the formation of a binary is studied in a series of two papers. This paper deals statistical theory with low initial velocities and equal masses in the evolution of stellar system in two-dimensional space. The complete statistical solutions (i.e. the distributions of eccentricity e of the binary, binary energy E_b and the escape velocity of escaper v_s ,etc.) of the system are calculated and are in good agreement with the numerical results of dynamical evolution in the range of [illegible] in all direction of in a plane. We have also applied double limit process to the system in a plane and observed that the perturbing velocity $v_0 \to 0^+$ the product of the semi-major axis 'a' of the final binary and the square of the escape velocity, i.e. [illegible] in all direction of α

1. Introduction:

We draw attention to areas of astrophysics where the three-body problem has been obtained a central position and also discuss some new areas where its importance has still to be proven. There has been tremendous progress in the understanding of the three-body problem since the large scale application of computer in orbital calculations began some fifty years ago. Even though it is still difficult to say exactly how a three-body system evolves from given initial conditions without numerical orbit calculations. It is now possible to make immediate qualitative statement as well as quantitative statement in dynamical and statistical sense as to what happens to the system in future.

One may divide the astrophysically interesting three-body systems into three main categories:

(i) Systems of stars (of comparable masses and initial distances) that break up after some dynamical evolution. Such configurations are typical in star forming regions.

(ii) Scattering of single star of close binaries. These events are important in the dynamics of star clusters.

(iii) Stable hierarchical systems. Examples of which are the numerous triple stars observed in the galactic field.

Here, we wish to move to the first category of three-body solution when the trajectories depend sensitively on the initial conditions.

The study of triple systems in comparison with a two-body systems is more useful, but it is far more difficult to study it analytically as well as numerically. The reason is that a two-body systems with negative total energy are always bounded while triple systems are not. It is convenient to study a three-body system by replacing it with two two-body systems. In this process, the original three-body system approaches its partition into two two-body systems, in which the escaper and the binary form a two two-body system and the members of the binary form another two-body system. In fact, the main problem is the partition of the phase space of the initial conditions. The region of phase space with bounded motion is mixed with escape regions according to Henon [9].

In this field, the dynamical evolution of triple encounters in the stellar system has attracted the attention of researchers for a long time. The conjecture of Birkhoff [5,6] and latter reformulated by Szebehely [17,18] known as Birkhoff-Szebehely conjecture states that sufficiently triple close simultaneously asymmetric approach results with the formation of a binary and a escape of the third body. It seems to be of fundamental importance in the global behavior of three gravitational interacting stars. The dynamical evolution of such system has been studied by many authors Agekian and Anogova [1], Szebehely [16,17,18], Anosova and Orlov [3], Valtonen [20], Valtonen and Mikkola [23], Chandra and Bhatnagar [7] and others.

Since Sundman [15] has shown that simultaneously close approaches occur only with small values of the total angular momentum C (in fact for triple collision C = 0 is a necessary condition) and study of system with low values of angular momentum C favour escape. The dynamical evolution of rotating triple system has been numerically studied by many authors (Anosova [27], Standish [28], Saslaw, Valtonen and Aarseth [24], Anosova, Bezotov and Orlov [2], Mikkola and Valtonen [10], Anosova and Orlov [4] etc.) and showed that the angular momentum is an important parameter in the description of the final state.

In addition to numerical dynamical evolutions, the dynamical disruption of triple system has been studied with statistical theories by so many authors; Heggie [8], Monaghan [11,12], Nash and Monaghan [13], Valtonen et al.[19] etc.

Thus, in recent years there have been two different approaches to the solving of the problem:

(i) A detail study of well specified case, and
(ii) A statistical study of large member of different systems.

Both methods have their advantage and disadvantage. Here, the subject is the general problem of three-body of the equilateral Lagrangian solution in symmetric rotating configuration. The masses of the participating bodies are taken equal. In Lagrangian solution, the symmetric configuration is never destroyed, therefore, escape does not occur and all motion are periodic, even when angular momentum C is small. If C is small and asymmetric changes of the initial conditions are introduced, it leads to escape instead of periodic orbits. The existence of such type of behavior are along the previously mentioned Birkhoff-Szebehely conjecture. For this, the triple close approach problem with two-parameter families has been studied by Chandra and Bhatnagar [7] (First Approach), in which, they have shown that how a symmetric triple collision is avoided by imparting very small velocity v_0 in a certain direction α. Our aim here is to study the problem analytical and precise numerical investigation in the framework of statistical escape theories (Second Approach) which is developed by Monaghan [11,12] and Valtonen [21]. This statistical theories of the disruption of triple system are based on the assumption that the phase trajectory of triple system is quasi-ergodic within the region of close triple approach. Then the probability of escape with certain orbital elements of the final binary and the escaping body is proportional to the corresponding volume of phase space in a coordinate system which is associated with the centre of mass of the triple system. According to Szebehely [25] and Agekian et al. [26] classification, the families presented in this paper belongs to the classification 'O'. Furthermore, our aim is also to study the problem with double limit-process.

2.Statement of the Problem

Three equal masses are considered to occupy initially the vertices of an equilateral triangle $P_1P_2P_3$. These particles is the absence of any disturbance will move along the medians of the triangle and collide at the centroid of the triangle. Here, to avoid such a collision the mass of P_3 is subjected to small perturbing velocity v_0 making an angle α with the line parallel to P_1P_2, the masses at P_1 and P_2 are also subjected to perturbing velocity $v_0/2$ parallel and opposite to v_0. The centre of mass of the system stays at the centroid of the triangle for all times.

Let there be three equal masses $m_1 = m_2 = m_3$ each equal to unity, occupy the vertices of an equilateral triangle $P_1P_2P_3$, the centroid 'O' as the origin and x-axis parallel to one of the sides P_1P_2 of the triangle. We also choose the unit of distance such that at $t = 0$, $P_1P_2 = P_2P_3 = P_3P_1 = 1$.

By symmetry each particle will be at the same distance from the origin. At t = 0, the position (x_i, y_i) and velocities $(\dot{x}_i, \dot{y}_i)$, i = 1,2,3 of P_1, P_2, P_3 are given by

$x_1 = 1/2, x_2 = -1/2, x_3 = 0,$

$y_1 = y_2 = -1/2\sqrt{3}, y_3 = 1/\sqrt{3};$

$\dot{x}_1 = \dot{x}_2 = -v_0 Cos\alpha/2; \qquad \dot{x}_3 = v_0 Cos\alpha,$

$\dot{y}_1 = \dot{y}_2 = -v_0 Sin\alpha/2, \ \dot{y}_3 = v_0 Sin\alpha.$

The introduction of non-zero initial velocities with any value of α break the symmetry and collision can be avoided. The distances between the masses are functions of v_0, α and t. Henceforth, the triangle $P_1P_2P_3$ cease to be an equilateral triangle.

Let r_{12}, r_{23}, r_{31} represent the lengths of the sides P_1P_2, P_2P_3, P_3P_1 respectively. These three distances are not equal any more. Chandra and Bhatnagar [7] has calculated the values of the relative distances up to the second order of time t as follows:

$$r_{12} = 1 - \frac{3}{2}t^2$$

$$r_{23} = r_{12} + \frac{3}{4}v_0\left(Cos\alpha + \sqrt{3}Sin\alpha\right)t + \frac{9}{8}v_0^2\left(1 - \frac{1}{4}\left(Cos\alpha + \sqrt{3}Sin\alpha\right)^2\right)t^2;$$

$$r_{31} = r_{32} - \frac{3}{4}v_0\left(Cos\alpha - \sqrt{3}Sin\alpha\right)t + \frac{9}{8}v_0^2\left(1 - \frac{1}{4}\left(Cos\alpha - \sqrt{3}Sin\alpha\right)^2\right)t^2.$$

It may be observed that r_{12} is independent of v_0 and α while r_{23} and r_{31} are functions of both to the order of t^2.

For the present study we require some more parameters at $t = 0$, which are given below:

(i) **Moment of Inertia I:**

$$I(t) = \sum_{i=1}^{3} m_i r_i^2 = 1 + \sqrt{3}\, v_0 \, Sin\alpha \; t + \frac{3}{2}\left(v_0^2 - 2\right)t^2 + O(t^3),$$

At $t = 0$,

$$I(0) = 1$$

$$\dot{I}(0) = \sqrt{3}\, v_0 \, Sin\alpha \; t,$$

$$\ddot{I}(0) = 3v_0^2 - 6.$$

(ii) **Total Energy E_t:**

$$E_t = K_t + V_t = \sum_{i=1}^{3} \frac{1}{2}(m_i \dot{r}_i^2) - \sum_{1\le i<j\le 3} \frac{m_i m_j}{r_{ij}}$$

At $t = 0$,

$$E_t = 3\left(\frac{v_0^2}{4} - 1\right),$$

where K_t and V_t are kinetic and potential energy of the system.

(iii) **Angular Momentum:**

$$C^2 = \left[\sum_{i=1}^{3} m_i r_i \times \dot{r}_i\right]^2.$$

$$\text{At } t = 0, \; |C| = \frac{\sqrt{3}}{2} v_0 Cos\alpha.$$

3.Possibility of Escape

Since the distance of the escaper from the two-body which form a binary will go on increasing and eventually will be greater than the relative distance between the binary, the body which is likely to escape must be opposite to the shortest distance between the participating bodies. Keeping this criteria in view, it may be observed from the relative distances of the participating bodies that when low value of v_0 and any value of α between 0^0 and 2π are introduced the initial symmetry gets destroyed and gives rise to escape of one of the bodies which is opposite to the smallest side provided escape conditions are satisfied.

In order to see which body is likely to escape and the other two forming a binary for low values of v_0 and different values of α are analysed on the basis of relative distances of the participating bodies. Our result agrees with Chandra and Bhatnagar [7]. The possibility of escape with the formation of binary in a plane are given in table-I.

TABLE-I

Possible region of escape with the formation of a binary for small value of V_0 and $0^0 \le \alpha \le 2\pi$.

Angle	Inequality of distances (t is sufficiently small)	Possibility of escape	Binary
$-\pi/2 < \alpha < \pi/6$	$r_{13} < r_{12} < r_{23}$ or $r_{13} < r_{23} < r_{12}$	m_2	m_1, m_3
$\pi/6 \le \alpha \le 5\pi/6$	$r_{12} < r_{13} < r_{23}$ or $r_{12} < r_{23} < r_{13}$	m_3	m_1, m_2
$5\pi/6 < \alpha < 3\pi/2$	$r_{23} < r_{12} < r_{13}$ or $r_{23} < r_{13} < r_{12}$	m_1	m_2, m_3
$\alpha = 3\pi/2$	$r_{23} = r_{13} < r_{12}$	Decision cannot be taken	

In this table-I, the inequalities in column 2 are true only for small values of t and corresponding to these inequalities we have mentioned the possibility of escape with the formation of a binary in column 3 and 4 respectively. The inequality may change for sufficiently large value of t for some values of α and the body that escapes with the formation of a binary may not be as mentioned in column 3 and 4, but the escaper must be opposite to the smallest relative distance of the participating bodies as we have stated earlier.

4. Analytical Theory

The three-body system approaches its partition into two two-body systems, in which the escaper and the centre of mass of the binary form a hyperbolic two-body system and the member of the binary form another an elliptic two-body system.

After the binary was formed, the total energy of the system is E_t may be written as

$$E_t = E_e + E_b \,, \qquad (1)$$

where, E_e is the escape energy, E_b is the stored energy in the binary. Equations for E_e and E_b may be written from two-body consideration as follows:

$$E_e = \frac{1}{2} M_2 \dot{r}_e^2 - G\frac{m_B m_e}{r_e} \,,$$

$$E_b = \frac{1}{2} M_1 \dot{r}^2 - G\frac{m_a m_b}{r} \,,$$

where m_e is mass of third body (escaper) and m_a and m_b are masses of the binary components. The position vector of third body (escaper) relative to the barycentre of the binary is r_e while the binary components are separated by the vector $r_{ij} = r$ ($1 \le i < j \le 3$). We call total mass of the binary components $m_B = m_a + m_b$ and total mass of the system $M = m_B + m_e$. Then the reduced masses $M_1 = \frac{m_a m_b}{m_B}$ and $M_2 = \frac{m_B m_e}{M}$.

Several escape conditions exist in the literature. For instance, Standish [14] has shown that it is sufficient for escape that

$$E_e \ge \frac{G m_1 m_2 m_3}{m_1 + m_3} \frac{d^2}{r_e^2 (r_e - d)} + \frac{M m_2}{2(m_1 + m_3)} v_e^2 \, \mathrm{Sin} \mu ,$$

and that $r_s > d = \frac{G}{|E_t|}\sum_{1\le i<j\le 3} m_i m_j$,

where μ is the angle between v_s and r_s.

These conditions are satisfied and escape does occur for sufficiently small values of v_0 and for all values of α in a plane. This follows from the fact that as $v_0 \to 0^+, I_m \to 0^+$, where, I_m = minimum moment of inertia. The asymmetric triple close approach, after reaching minimum moment of inertia I_m generates sufficiently large values of I and $\dot{I}$ for escape. For an indication of this process see Birkhoff [6] and Szebehely [17]. Chandra and Bhatnagar [7] have followed Sundman [15] escape condition.

According to the above escape condition, m_1, m_3 form a binary and m_2 escapes. It has to be modified according to the binary components and escaper (Table I).

In our case, the total mass of the system M = 3, the total mass of the binary $m_B = 2$ and the reduced masses $M_1 = 1/2$ and $M_2 = 2/3$.

(I) Statistical Process

(A) Phase Space Volume:

In the statistical theory of the disintegration of three-body systems by Monaghan's [11,12] and Valtonen [21], it is assumed that the probability of a given escape configuration is proportional to the volume in phase space available to this configuration. This assumption is supported by results from other theoretical approaches as well as from numerical orbit calculations.

The density σ of states (escape configuration in the phase space per unit energy) is obtained by integrating δ-function over the phase space volume with the phase space co-ordinates r, r_s, p, p_s where p and p_s are the canonical momenta associated with r and r_s respectively.

$$\sigma = \iiiint \delta\left(\frac{P_s^2}{2} + V_s + E_b - E_t\right) dr_s, dp_s, dr, dp \qquad (2)$$

Where we put

$$E_s = \frac{P_s^2}{2M_2} + V_s, \left(P_s = \frac{2}{3}\dot{r}_s\right).$$

To determine the properties of the binary, we integrate eqn. (2) over r_s and p_s and for escaper we integrate over r and p. Firstly we carry out the integrations over the momentum space p_s with a uniform distribution of directions over the whole sphere. From the rules for integrating δ-functions

$$\sigma = \int \delta\left(\frac{P_s^2}{2M_2} + V_s + E_b - E_t\right) dp_s$$

$$= 2\pi\int_0^\infty \delta\left(\frac{P_s^2}{2M_2} + V_s + E_b - E_t\right) P_s dP_s$$

$$= 2\pi M_2 \int_0^\infty \delta\left(x - (E_t - V_s - E_b)\right) dx$$

$$= 2\pi M_2 \qquad (3)$$

Where $x = P_s^2/2M_2$ and from the property of the δ-function:

$$\int_0^\infty f(x)\delta(x-a)dx = f(a) = 1.$$

Monaghan [11] integrates over r_e and p_e, assuming that the escaper orbit is radially outward from the barycentre of the binary while Valtonen [21] assume that the escaper at a distance r_e from the binary centre has come to this point along a straight line orbit. It must have come from the neighbourhood of the binary. Otherwise it would not have acquired the escape velocity. The neighbourhood of the binary may be defined for our purposes as a circular area, perpendicular to the vector over r_e, with the radius of some simple multiple na, where a is the semi-major axis of the binary and the multiple n comes from experiences with orbit calculations and is ≈ 7. The semi - major axis 'a' of the binary is related to the binary energy by

$$a = -\frac{m_a m_b}{2E_b} . \qquad (4)$$

The straight line drawn from the point r_e through the circle of radius 7a define a cone; this is called the "loss cone" because particle travelling in reverse direction from the apex of the cone to the binary will generally be scattered away from the cone and thus these orbits are lost. In the current problem we can say that only the orbits within the loss cone are true escape orbits. Since they have been strongly influenced by the binary in the past. The loss cone direction contain approximately the fraction of $\pi(7a)/2\pi r_e$ with radius r_e surrounding the escaper. Since δ - functions does not depend on r_e. We can immediately perform the integration taking account of the loss cone factor $\pi(7a)/2\pi r_e$ with $dr_e = 2\pi r_e dr_e$ and 2nd integral becomes

$$2\pi\int_0^R \frac{14a}{2\pi r_e} r_e dr_e = 14a R$$

$$= 7\frac{Gm_a m_b}{|E_b|} R , \quad \left(a = \frac{Gm_a m_b}{2|E_b|}\right).$$

Where the upper limit R of the r_e range is considered a free parameter. This leads to

$$\sigma = 14\pi M_a G m_a m_b R \int \frac{1}{|E_b|} dr\, dp , \qquad (5)$$

where $\left(P = \frac{1}{2}\dot{r}\right)$.

In order to see the significance of the remaining integrals, we use polar co-ordinates (r, θ) and write the remaining integral in the form

$$\int dr\, dp = \iint\iint dr\, dp_r\, d\theta\, dp_\theta . \qquad (6)$$

With the change of variable

$$E_b = \frac{1}{2}\frac{P^2}{M_1} - \frac{Gm_a m_b}{r}$$

$$= \frac{1}{2M_1}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{Gm_a m_b}{r}$$

From which

$$\frac{dE_b}{dp_r} = \frac{1}{M_1} p_r, \text{ or } dp_r = \frac{M_1 dE_b}{p_r} .$$

Therefore

$$p_r = \left[2M_1E_b - \frac{1}{r^2}p_\theta^2 + \frac{2M_1Gm_a m_b}{r}\right]^{1/2}$$

Where we put

$$x = r, L = p_\theta ,$$

from which, we get

$$p_r = \left[2M_1E_b - \frac{1}{x^2}L^2 + \frac{2M_1Gm_am_b}{x}\right]^{1/2}$$

The integral in eqn (6) becomes

$$\iint \frac{dE_b}{|E_b|} \frac{x\,M_1\,dL\,dx\,d\theta}{\left[2M_1E_bx^2 - L^2 + 2Gm_am_bM_1x\right]^{1/2}}$$

(7)

The integration over x can be written as

$$\int_{x_1}^{x_2} \frac{x\,dx}{\left[-L^2 + 2Gm_am_bM_1x - 2M_1|E_b|x^2\right]^{1/2}}$$

and is evaluated on noting that the limits are determined by the zeros of the denominator and both limits x_1, x_2 are less than R for the possible choice of this parameter. After evaluation total contribution of the integration over x becomes

$$\frac{\pi}{2\sqrt{2}} \frac{Gm_am_b}{|E_b|^{3/2} M_1^{1/2}}.$$

To integrate L it is more useful to transform to the eccentricity e with the binary orbit formula

$$L^2 = M_1 \frac{(Gm_am_b)^2}{2|E_b|^{1/2}}(1-e^2).$$

Combining the integrations over x, L and θ (put $\theta = \frac{\pi}{2}$) together with eqn. (5), we find the density of states

$$\sigma = 7\pi^3 M_1M_2R(Gm_am_b)^3 \iint \frac{dE_b}{|E_b|^5} \frac{e\,de}{\sqrt{(1-e^2)}}$$

(8)

It may be noted here that we consider only the quantities which follow the integral signs and leave the coefficients in front of them. These quantities represent the distribution over which one has to integrate in order to obtain the total phase space volume. These are the fundamental distribution in which we are interested. Thus the distribution of binary energy $|E_b|$, normalized to unity, is

$$f(|E_b|)\,d|E_b| = 2|E_i|^2|E_b|^{-3}\,d|E_b| \qquad (9)$$

This result may also be derived from other theoretical concept like Heggie [8] and from such concept, equation (9) represents an equilibrium distribution.

The corresponding distribution of eccentricity is

$$f(e)de = e(1-e^2)^{-1/2}de. \qquad (10)$$

The statistical theory therefore predicts that there is no correlation between the binary energy E_b and eccentricity e of the binary. Since the semi-major axis a is proportional to $1/E_b$ an equivalent prediction is that a and e are independently distributed.

(B) Escape Velocity:

When the binary energy E_b is known, then escape energy $E_s = |E_b| - |E_i|$ is obtained from the equation (1). We substitute this in equation (8) and integrate over e, the equation (8) becomes

$$\sigma = 7\pi^3 M_1M_2R(Gm_am_b)^3 \iint \frac{dE_s}{(|E_i| + |E_s|)^5}.$$

(11)

We assume that the velocity of the escaper in the centre of mass coordinate system is v_s. Then

$$v_s = \frac{m_B}{M}|\dot{r}_s|, \text{ and}$$

$$E_s = \frac{1}{2}M_s|\dot{r}_s|^2 = \frac{1}{2}\frac{m_s M}{m_B}v_s^2.$$

Therefore $dE_s = \frac{m_s M}{m_B}v_s dv_s$, and equation (11) becomes

$$\sigma = 7\pi^2 M_l R m_s^2 (G m_s m_b)^5. \quad (12)$$

$$\int \frac{v_s dv_s}{\left(|E_t| + \frac{1}{2}(m_s M/m_B)v_s^2\right)^9}$$

From the above, the escape velocity can be written as $v_s = \left(\frac{2m_B E_s}{m_s M}\right)^{1/2}$.

In our case, this escape velocity becomes

$$v_s = \frac{2\sqrt{E_s}}{\sqrt{3}}, \quad (13)$$

which depends on the escape energy and escape energy depends on both the parameters v_0 and α.

The escape velocity distribution after normalization is

$$f(v_s)dv_s = \frac{(2|E_t|^2 m_s M/m_B)v_s dv_s}{\left(|E_t| + \frac{1}{2}(m_s M/m_B)v_s^2\right)^9}, \quad (14)$$

and the peak of this distribution is

$$(v_s)_{peak} = \frac{1}{2}\sqrt{\frac{(M-m_s)}{m_s M}}\sqrt{|E_t|}, \quad (15)$$

when putting $df(v_s)/dv_s = 0$.

In our case, the escape velocity distribution is

$$f(v_s)\, dv_s = \frac{27\, v_s dv_s}{\left(3 + \frac{3}{4}v_s^2\right)^9},$$

and the peak of the escape velocity distribution is

$$(v_s)_{peak} = \frac{2}{\sqrt{15}}\sqrt{|E_t|} = 0.894427191.$$

(I I) Double Limit Process:

The asymmetric triple close approach results in an escaper with the formation of a binary. The (asymptotic) hyperbolic escape velocity relative to the centre of mass of the system, v_s and the asymptotic value of the semi-major axis of the binary 'a' depend on the perturbation measured by v_0. As $v_0 \to 0^+$, whatever direction may be, the product

$$a\, v_s^2 \to \frac{2}{3}.$$

To establish such concept, first of all, attention is directed to the double limit process involved in the above result. In this limit process, the original three-body problem approaches its partition into two two-body problems. The escaper and the centre of mass of the binary form a hyperbolic two-body problem and the members of the binary form an elliptic two-body problem. After the binary is formed, the distance between the escaper and the binary, $r_s \to \infty$, the velocity of the escaper $v_s \to v_\infty$.

The second limit process refers to the behavior of the members of the three-parameter families as the perturbing initial velocity $v_0 \to 0^+$, whatever direction may be. Thus the above result may be written as

$$a v_\infty^2 \to \frac{2}{3}.$$

We have total energy of the system,

$$E_t = E_b + E_s.$$

In the first limit process E_t is fixed, $E_b \to -G\frac{m_a m_b}{2a}$, and $E_s \to \frac{1}{2}\frac{m_s M}{m_B} v_s^2$,

$$\text{Therefore, } E_t = \frac{1}{2}\frac{m_s M}{m_B} v_\infty^2 - \frac{G m_a m_b}{2a}.$$

In the second limit process, we consider the general form of the total energy

$$E_t = \frac{3}{4} m v_0^2 - \frac{3Gm^2}{l},$$

where $m = m_1 = m_2 = m_3$ and l is the length of the side of the equilateral triangle at the beginning of the motion.

Equating these two forms of the above total energy, we have

$$a v_\infty^2 = G\frac{m_a m_b}{m_s}\left(\frac{m_B}{M}\right) + \frac{3}{2}a\left(\frac{m m_B}{M m_s}\right).\left(v_0^2 - 4\frac{Gm}{l}\right)$$

As $v_0 \to 0^+$, the following limiting values are obtained:

Minimum moment of inertia $I_m \to 0^+$, the angular momentum $C \to 0^+$, the total energy $E_t \to -3Gm^2 l^{-1}$, the semi-major axis $a \to 0^+$ and the escape velocity $v_\infty \to \infty$, and consequently,

$$a v_\infty^2 \to G\frac{m_a m_b}{m_s}\left(\frac{m_B}{M}\right).$$

In our case, $a v_\infty^2 \to \frac{2}{3}$.

It may noted that if the above simplifying substitution is made prior to the second limit process, we have

$$a\left(v_\infty^2 - v_0^2 + 4\right) = \frac{2}{3}, \text{ or}$$

$$v_\infty^2 = \left(v_0^2 + 2(3a)^{-1} - 4\right),$$

allowing an orderly presentation of numerically established members of the family.

It is essential to point out that the process allows the generation of arbitrary high-escape velocities with arbitrary close binaries.

5. Numerical Analysis

As the purpose of this paper is to investigate the results of triple close approaches with the formation of binary and the systematic regularity of escape of the third body in the framework of statistical escape theories. For this, we have studied the problem with small perturbing velocities v_0 for $10^{-10} \le v_0 \le 10^{-1}$ and for all values of α i.e. $0° \le \alpha \le 2\pi$.

Chandra and Bhatnagar [7] has observed in actual experiment that two close approaches occur before the formation of a binary between the participating bodies. The first close approach means the first minimum smallest relative distance r_{ij} between the participating bodies. This occurs before minimum moment of inertia I_m is attained. And the body which actually escape is opposite to this smallest relative distance r_{ij}. The second close approach means the smallest relative distance between the participating bodies when I_m is attained. This minimum moment of inertia I_m is attained (second close approach) slightly latter than the first close approach. The two bodies amongst the participating bodies which has the first close approach are different from the two bodies which has the second close approach except for special regions. In special regions, bodies are the same and the first close approach occurs very slightly before attaining of I_m in the neighbourhoods of $\alpha = 30^o, \alpha = 150^o$ and $\alpha = 270^o$. It may be seen in Table II (Chandra and Bhatnagar [7]). It may be noted for $\alpha = 3\pi/2$, it is found in actual experiment that m_3 escape and m_1, m_2 formed binary. The total energy of the system is -3.

Since triple close approaches results with the formation of a binary and escape of the third body when minimum moment of inertia I_m is attained. So, we have carried out experiment according to Chandra and Bhatnagar [7,sec.3] to obtain the definite orbital elements of the final binary and the escaper when minimum moment of inertia I_m is attained for $10^{-10} \le v_0 \le 10^{-1}$ and $-\pi/2 \le \alpha \le \pi/2$. For this, we have obtained semi-major axis 'a' from the maximum and minimum distances r_{ij} of the binary. Suffix i and j of r are according to Table I. When semi-major axis a is obtained from equation (4) then from equation (1), we have calculated binary energy E_b and escape distances of the binary against each angle α.

energy E_s respectively. We have also calculated eccentricity e, distribution of eccentricity $f(e)$, the magnitude of escape velocity v_s, distribution of escape velocity $f(v_s)$, distribution of $|E_b|$ i.e. $f(|E_b|)$. We have also adopted double limit process and calculated escape velocity which is denoted by v_∞ and it gives promising value as the value of escape velocity v_s (i.e. $v_s \equiv v_\infty$).

It is also essential that the total energy $E_t < 0$, therefore $E_s = E_t - E_b = |E_b| - |E_t|$, for escape $|E_b| > |E_t|$ is required. This is always established with sufficiently small value of semi-major axis a. Therefore escape for negative total energy is associated with the formation of a close binary.

We define the following two families with the parameters v_0 and α,

(i) For first family, varying α and keeping v_0 fixed,

(ii) For second family, varying v_0 and keeping α fixed.

As our main aim is to study the effect of α with v_0, so we have studied the typical representative member of the family. For this we have taken $v_0 = 10^{-9}$ and α lying between $-\pi/2$ and $\pi/2$ at an interval of 5^o (Table II). The role of m_1 and m_3 are interchanged when α is replaced by $(180^o - \alpha)$. The relative distances of the participating bodies are also interchanged accordingly. For $\alpha = \pm 90^o$, the system remains symmetric about the y-axis.

In the columns of $r_{ij}(max)$ and $r_{ij}(min)$ of table II are maximum and minimum

Table-II

Binary distance R_{ij}(max.) and R_{ij}(min.), a, E_b, E_s, e, binary and escaper for $-\pi/2 \le \alpha \le \pi/2$, when $v_0 = 10^{-3}$.

Ang. α↓	Rij(max.)	Rij(min.)	Semi-maj axis a	Binary Energy E_b	Escape Energy E_s	Eccentricity e	Binary	Esca-per
-90	5.644578E-03	2.835708E-05	2.835708E-03	-176.947	173.947	0.999123	m_1,m_3	m_2
-85	2.881826E-03	2.399868E-03	1.564148E-03	-31.966	28.966	0.840582	m_1,m_3	m_2
-80	5.836168E-03	1.032448E-03	3.434308E-03	-145.598	141.598	0.699574	m_1,m_3	m_2
-75	2.762578E-03	1.348978E-03	2.055378E-03	-243.514	240.514	0.345449	m_1,m_3	m_2
-70	2.048988E-03	1.407728E-03	1.728358E-03	-289.293	286.293	0.185512	m_1,m_3	m_2
-65	2.380178E-03	9.480338E-04	1.664408E-03	-300.409	297.409	0.430858	m_1,m_3	m_2
-60	2.760978E-03	6.119148E-04	1.686448E-03	-296.482	293.482	0.637157	m_1,m_3	m_2
-55	3.083938E-03	3.850188E-04	1.734478E-03	-288.272	285.272	0.778028	m_1,m_3	m_2
-50	3.349758E-03	2.275678E-04	1.788868E-03	-279.539	276.539	0.872772	m_1,m_3	m_2
-45	3.546338E-03	1.190068E-04	1.832628E-03	-272.685	269.685	0.935298	m_1,m_3	m_2
-40	3.696928E-03	4.854728E-05	1.850898E-03	-270.141	267.141	0.997377	m_1,m_3	m_2
-35	3.774188E-03	9.995008E-06	1.892148E-03	-264.251	261.251	0.994718	m_1,m_3	m_2
-30	3.794678E-03	3.389788E-07	1.897508E-03	-263.504	260.505	0.999822	m_1,m_3	m_2
-25	3.761628E-03	1.888308E-05	1.890158E-03	-264.528	261.528	0.990116	m_1,m_3	m_2
-20	3.658958E-03	6.608058E-05	1.862528E-03	-268.454	265.454	0.964521	m_1,m_3	m_2
-15	3.504178E-03	1.458728E-04	1.824928E-03	-273.984	270.984	0.920176	m_1,m_3	m_2
-10	3.290308E-03	2.638868E-04	1.778148E-03	-281.192	278.192	0.850414	m_1,m_3	m_2
-5	3.004598E-03	4.363588E-04	1.720478E-03	-290.618	287.618	0.746373	m_1,m_3	m_2
0	2.677718E-03	6.716028E-04	1.674668E-03	-298.569	295.569	0.598961	m_1,m_3	m_2
5	2.308978E-03	1.012488E-03	1.658338E-03	-301.527	298.527	0.389419	m_1,m_3	m_2
10	2.004008E-03	1.447008E-03	1.725508E-03	-289.771	286.771	0.161402	m_1,m_3	m_2
15	3.376508E-03	1.247008E-03	2.311758E-03	-216.286	213.286	0.460852	m_1,m_3	m_2
20	4.748008E-03	1.047008E-03	2.898008E-03	-172.533	169.533	0.638716	m_1,m_3	m_2
25	9.934828E-03	1.085498E-03	5.510308E-03	-90.743	87.742	0.881187	m_1,m_3	m_2
30	1.105008E-01	7.206008E-03	5.885308E-03	-8.496	5.4957	0.877558	m_2,m_3	m_1
35	5.259008E-03	6.673008E-04	2.662808E-03	-18.777	15.777	0.974941	m_1,m_3	m_2
40	6.307008E-03	1.040128E-03	3.673508E-03	-136.110	133.110	0.716891	m_1,m_3	m_2
45	2.768948E-03	1.354828E-03	2.059408E-03	-242.792	239.792	0.342122	m_1,m_3	m_2
50	2.060788E-03	1.398308E-03	1.728508E-03	-288.104	285.104	0.191549	m_1,m_3	m_2
55	2.397628E-03	9.391318E-04	1.668408E-03	-299.695	296.695	0.437089	m_1,m_3	m_2
60	2.765368E-03	6.134658E-04	1.688908E-03	-296.057	293.057	0.637351	m_1,m_3	m_2
65	3.075018E-03	3.934128E-04	1.733708E-03	-288.399	285.399	0.773658	m_1,m_3	m_2
70	3.338748E-03	2.342448E-04	1.786508E-03	-279.878	276.878	0.868888	m_1,m_3	m_2
75	3.545088E-03	1.260418E-04	1.835508E-03	-272.400	269.400	0.931333	m_1,m_3	m_2
80	3.685578E-03	5.437088E-05	1.868908E-03	-267.384	264.384	0.970825	m_1,m_3	m_2
85	3.794818E-03	1.336318E-05	1.893508E-03	-264.055	261.055	0.992843	m_1,m_3	m_2
90	3.847088E-03	1.934508E-06	1.934508E-03	-259.807	256.807	0.999012	m_1,m_2	m_3

In the columns of r_{ij}(max) and r_{ij}(min) of table II are maximum and minimum distances of the binary against each angle α. These maximum and minimum distances are also known as Apastron and Peristron. Corresponding to these columns, the other columns give the values of semi-major axis a, binary energy E_b, escape energy E_s, eccentricity e. The member of binary and escaping body are mentioned in last two columns of table II.

Table-III

istribution of $f(E_b, l)$ and distribution of escape velocity $f(v_e)$ for $10^{-10} \leq v_0 \leq 10^{-1}$, when $\alpha = 10^{0}$.

$v_0 \rightarrow$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$f(E_b, l)$	8.312E-03	2.463E-04	7.358E-07	[illegible]	4.718E-12	1.213E-12	2.578E-17	7.148E-20	1.771E-22	4.398E-23
$f(v_e)$	1.098E-03	2.463E-03	3.170E-05	1.500E-07	1.828E-09	3.282E-10	4.610E-14	3.108E-16	2.098E-18	1.413E-20

he following characteristics of the above nentioned families are observed that

(I) The Distribution of Eccentricity f(e) and e^2

he distribution of eccentricities f(e) in the reakup of planar three-body systems for oth the family are prominent with umerical results. Since the factor $(1-e^2)^{-1/2}$ ppears in equation (10), so the haracteristic of f(e) with respect to e^2 may e observed for both the family. For the first amily, it is observed that f(e) fluctuates etween maximum and minimum values and ltimately rise towards $e \rightarrow 1$ that is very rominent ($e<1$). It may be seen in figure a. And for the second family, it fluctuate up o a certain value of v_0 and then increases vith e^2 (Fig. 1b.) and ultimately it also rise o $e \rightarrow 1$.

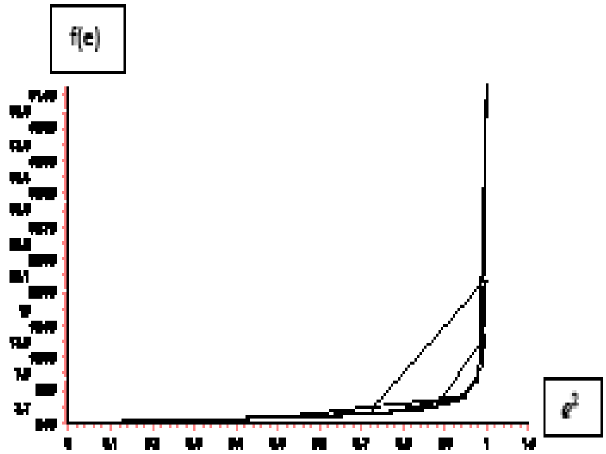

Fig. 1a. Eccentricity distributions f(e) vs. e^2

$(-\pi/2 \leq \alpha \leq \pi/2,\ v_0 = 10^{-4})$.

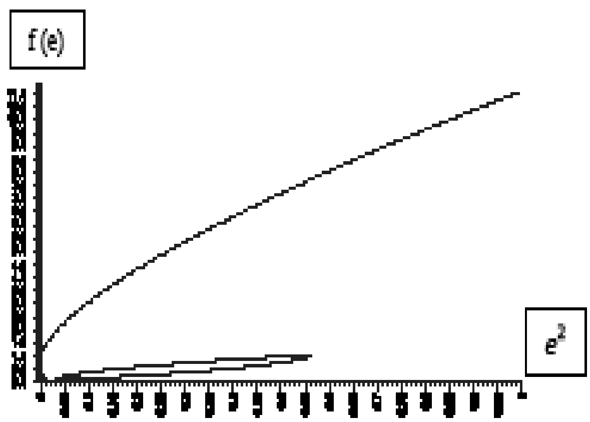

Fig. 1b. Eccentricity distributions f(e) vs. e^2

$(10^{-10} \leq v_0 \leq 10^{-1},\ \alpha = 10^{0})$.

(II) The Distribution of $|E_b|$ i.e. $f(|E_b|)$

It is observed from fig.2. for the first family that the binary energy distribution $f(|E_b|)$ fluctuates between maximum and minimum value and get maximum pick point at the neighbourhood of $\alpha = 30^0$. And for the second family, it may observed from table III that the value of $f(|E_b|)$ almost increases with v_0 .

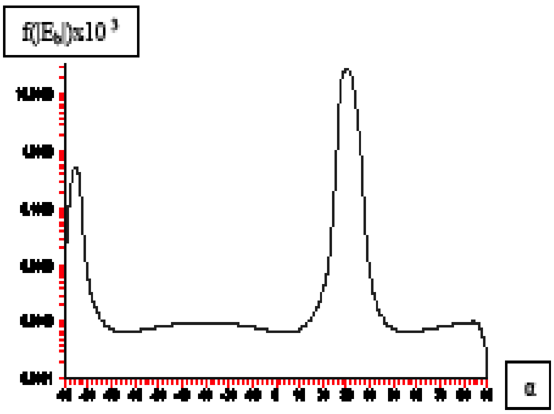

Fig.2. Binary energy distributions $f(|E_b|)$ vs. α

$(-\pi/2 \le \alpha \le \pi/2 , v_0 = 10^{-3})$.

(III) The Escape Velocity v_s and its Distribution $f(v_s)$:

It is observed from fig. 3a, for the first family that the escape velocity v_s and its distribution $f(v_s)$ fluctuate between maximum and minimum values. It may also be observed from this figure that v_s has minimum value at the neighbourhood of $\alpha = 30^0$ where as $f(v_s)$ has maximum value at that neighbourhood. For the second family, it is observed from fig.4b. that v_s increases or decreases as v_0 decreases or increases. And for the second family of escape velocity distribution $f(v_s)$, it may be observed from table III that it increases with v_0 increases. It may also be observed from fig.3b. that $f(v_s)$ fluctuates between maximum and minimum values with respect to v_s for the first family. For the second family, the character of $f(v_s)$ with respect to v_s may be same.

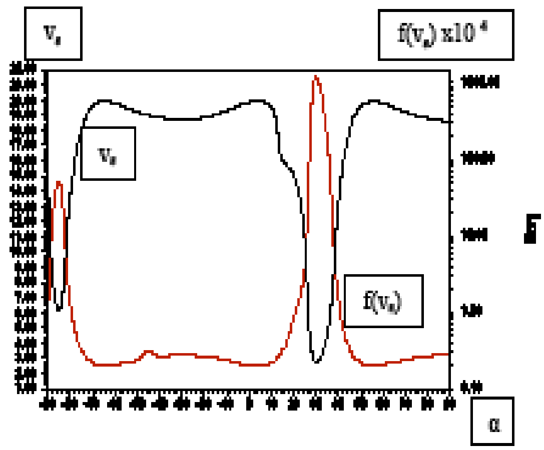

Fig.3a. Escape velocity distributions $f(v_s)$ & v_s versus. α $(-\pi/2 \le \alpha \le \pi/2 , v_0 = 10^{-3})$.

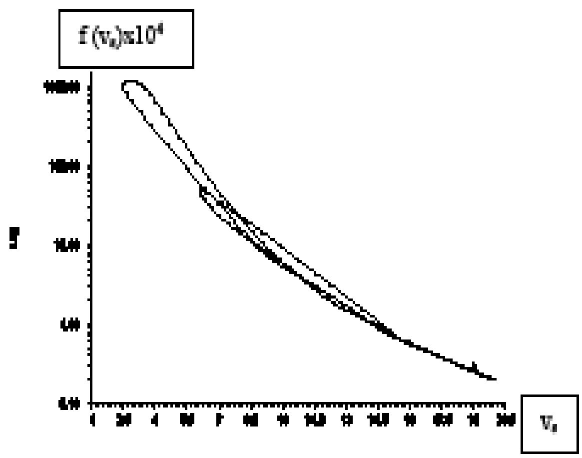

Fig.3b. Escape velocity distributions $f(v_s)$ vs. v_s

$(-\pi/2 \le \alpha \le \pi/2 , v_0 = 10^{-3})$.

(IV) In Double Limit Process

it is observed from numerical experiment that v_∞ and v_1 are in good agreement (i.e. $v_\infty \cong v_1$). It may be observed from fig.4a for the first family that $A = a v_\infty^2$ fluctuates between maximum and minimum values after 4 decimal places and ultimately $A \to 2/3$. And for the second family, A decreases with v_0 increases up to a certain value of v_0 and then increases and ultimately $A \to 2/3$. It may be seen in fig.4b. This support our above statement that $a v_\infty^2 \to 2/3$ when $v_0 \to 0^+$ in all direction of α.

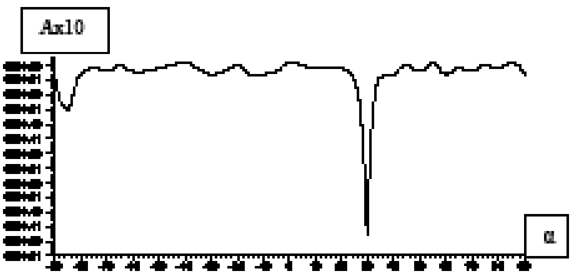

Fig.4a. A $(= a v_\infty^2)$ versus α $(-\pi/2 \le \alpha \le \pi/2,\ v_0 = 10^{-6})$.

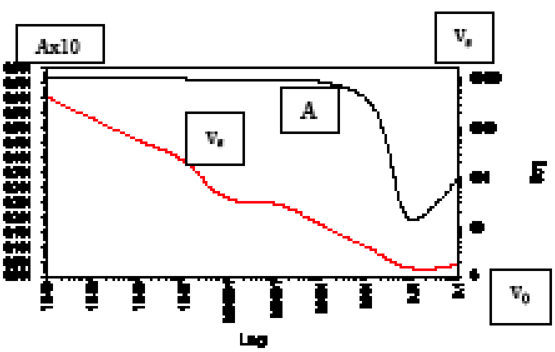

Fig.4b. A $(= a v_\infty^2)$ & v_1 versus v_0 ($\alpha = 10^\circ$).

Conclusion

The dynamical escape with the formation of a binary from a triple systems has been studied in the frame work of statistical escape theories in a plane. We have used ergodic principle as a useful tool in the description of the statistical results of the three-body break-up. This work is based on the theory of Monaghan and later reformulated by Valtonen and associates, and its general principles are confirmed in good agreement with numerical results of dynamical evolution in two dimensional space. We have also demonstrated that the double limit process is a useful tool to describe the three-body break-up.

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