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Equitable Cototal Domination in Graphs

Mathematics

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ABSTRACT

Let G=(V,E) be a simple graph. A subset D of V is called an equitable dominating set if for every v∈V-D, there exists a vertex u∈D such that uv∈E(G) and |deg(u)-deg(v)|≤1, where deg(u) and deg(v) denotes the degree of a vertex u and degree of a vertex v respectively. An equitable dominating set D is said to be an equitable cototal dominating set (ecd-set) if the induced subgraph ⟨V-D⟩ has no isolated vertices. The equitable cototal domination number $\gamma^e ct$ (G) of a graph G is the minimum cardinality of an ecd-set of G. In this paper, we obtain some bounds on $\gamma^e ct$ (G) and its exact values for some standard class of graphs are found.

1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. The order and size of G are denoted by p and q respectively. The vertex $v \in V$ is called a *pendant vertex* if $deg_G(v) = 1$ and an *isolated vertex* if $deg_G(v) = 0$, where $deg_G(x)$ is the degree of a vertex $x \in V$. We denote $\delta(G)(\Delta(G))$ as the *minimum* (*maximum*) degree. As usual, $\lceil x \rceil$ ($\lfloor x \rfloor$) be the smallest integer not less than x (be the greatest integer not greater than$x$). In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X.

For terminology and notations not specifically defined here, we refer to [2]. For more details about domination number and its related parameters, we refer to [3 and 5].

A set $D \subseteq V$ of a graph $G = (V, E)$ is a *dominating set* of G if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that v is adjacent to u. A dominating set D is said to be *minimal* if no proper subset of D is a dominating set. The minimum cardinality of a minimal dominating set of G is called *domination number* $\gamma(G)$ of G. A dominating set D is said to be *cototal dominating set* if $\langle V - D \rangle$ has no isolated vertices. The *cototal domination number* $\gamma_{ct}(G)$ of G is the minimum cardinality of a minimal cototal dominating set of G. This concept was introduced by Kulli et.al [6].

A subset D of V is called an *equitable dominating set* of a graph G if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma^e(G)$ and is called an *equitable domination number* of G. D is minimal if for any vertex $u \in D$, $D - \{u\}$ is not an equitable dominating set of G. This concept was introduced by Swaminathan et.al [8].

In this paper, our aim is to introduce a new domination parameter in the field of domination in theory of graphs which is as follows:

An equitable dominating set D of a graph G is said to be an *equitable cototal dominating set* (*ecd*-set), if $\langle V - D \rangle$ has no isolated vertices. An *ecd*-set is said to be *minimal* if no proper subset of D is an *ecd*-set. The *equitable cototal domination number* $\gamma^e_{ct}(G)$ of G is the minimum cardinality of minimal *ecd*-set of G.

Example:

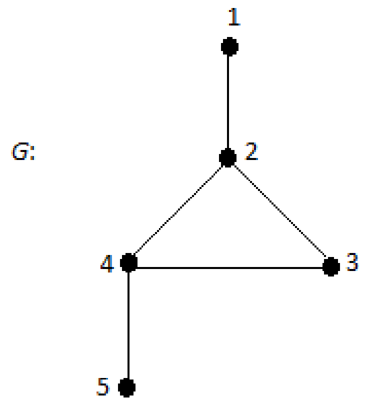

Fig 1.

In Fig 1, $V(G) = \{1, 2, 3, 4, 5\}$

The minimal equitable dominating sets are: $D_1 = \{1, 2, 5\}$, $D_2 = \{1, 3, 5\}$,

$D_3 = \{1, 4, 5\}$ etc. Therefore $\gamma^e(G) = |D_1| = |D_2| = |D_3| = 3$.

The minimal equitable total dominating set is: $D_1 = \{1, 2, 4, 5\}$.

Therefore $\gamma_t^e(G) = |D_1| = 4$.

The minimal equitable cototal dominating sets are: $D_1 = \{1, 2, 5\}$, $D_2 = \{1, 3, 5\}$, $D_3 = \{1, 4, 5\}$ etc.

Therefore $\gamma_{ct}^e(G) = |D_1| = |D_2| = |D_3| = 3$.

Remark 1.1. *For any graph G,*

$\gamma(G) \leq \gamma^e(G) \leq \gamma_t^e(G) \leq \gamma_{ct}^e(G)$.

The following observations are immediate.

Observations: For any connected graph G,

(i) $\gamma(G) \leq \gamma_t^e(G)$ and $\gamma_t^e(G) \leq \gamma_{ct}^e(G)$.

(ii) $\gamma_t^e(G) = \gamma_{ct}^e(G)$ if and only if G is either $K_{1,p}; p \geq 2$ or $K_{m,n}$; $m, n \geq 2$.

(iii) If D is a γ_{ct}^e − set of G, then $\langle V - D \rangle$ is a tree.

2. Results

We begin our investigation of the equitable cototal domination number by computing its values for several well known classes of graphs. In several instances, we shall have causes to use the ceiling function of a number.

Next, we list the exact values of $\gamma_{ct}^e(G)$ for some standard class of graphs.

Proposition 2.1.

1. For any path P_p ; $p = 3r + k$, where r is an integer and $k \geq 0$,

$$\gamma_{ct}^e(P_p) = \begin{cases} r+1; & if\ k = 1 \\ r+2; & if\ k = 0,\ 2. \end{cases}$$

2. For any cycle C_p with $p \geq 3$ vertices,

$$\gamma_{ct}^e(C_p) = \begin{cases} p-2; & if\ 3 \leq p \leq 5 \\ \frac{p}{3}; & if\ p \equiv 0\ (mod\ 3) \\ \lfloor \frac{p}{4} \rfloor + 2; & if\ p \equiv 0\ (mod\ 3). \end{cases}$$

3. For any complete graph K_p with $p \geq 3$ vertices, $\gamma_{ct}^e(K_p) = 1$.

4. For any star $K_{1,p}$ with $p \geq 3$ vertices, $\gamma^e_{ct}(K_{1,p}) = p$.

5. For any wheel W_p with $p \geq 4$ vertices, $\gamma^e_{ct}(W_p) =$

$$\begin{cases} 1; & if\ p = 4,\ 5 \\ \lceil \frac{p}{3} \rceil + 2; & if\ p \equiv 0\ (mod\ 3) \\ \lfloor \frac{p-2}{3} \rfloor + 2; & if\ p \equiv 0\ (mod\ 3)\,. \end{cases}$$

6. For any complete bipartite graph $K_{m,n}$, $\gamma^e_{ct}(K_{m,n}) =$

$$\begin{cases} 2; & if\ |m-n| \leq 1,\ 1 \leq m \leq n \\ m+n; & if\ |m-n| \geq 2,\ m,\ n \geq 2\,. \end{cases}$$

Next Theorem is immediate from Proposition 2.1.

Theorem 2.2. *Let G be a nontrivial connected graph. Then* $1 \leq \gamma^e_{ct}(G) \leq p$ *and these bounds are sharp. Further, the equality of lower bound is attained if and only if G is* $K_p; p \geq 2$ *and the equality of an upper bound attained if and only if G is* $K_{1,p}; p \geq 3$.

If G is said to be a $(k, k+1)$ *bi-regular graph*, then the degree of each vertex in G is either k or $k+1$, where k is a positive integer.

Theorem 2.3. *If* G *is a* $(r,\ r+1)$ *bi-regular graph for some* r, *then* $\gamma_{ct}(G) = \gamma^e_{ct}(G)$.

Proof. Suppose G is a bi-regular graph. Then every vertex of G has either r or $r+1$, where r is a positive integer. Let D be a minimal cototal dominating set of G. Then $|D| = \gamma_{ct}(G)$. Let $u \in V - D$, then there exists a vertex $v \in D$ such that $uv \in E(G)$, $deg(u) = r$ or $r+1$ and $deg(v) = r$ or $r+1$. Therefore $|deg(u) - deg(v)| = 0$ or 1. Therefore D is an equitable cototal dominating set of G. Hence $\gamma^e_{ct}(G) \leq |D| = \gamma_{ct}(G)$. But from observation(i), $\gamma_{ct}(G) \leq \gamma^e_{ct}(G)$. Therefore $\gamma_{ct}(G) = \gamma^e_{ct}(G)$.

Theorem 2.4. *For any connected graph G of order p containing a cycle.* $\gamma^e_{ct}(G) = p - 2$ *if and only if* $G = C_p$; $3 \leq p \leq 5$ *or G can be obtained from* $C_p; p \leq 5$ *by attaching zero or one pendant vertex to at most p vertices of the cycle.*

Proof. If $G = C_p$; $3 \leq p \leq 5$ or G can be obtained from $C_p; p \leq 5$ by attaching zero or one pendant vertex to at most p vertices of the cycle, then it is easy to verify that $\gamma^e_{ct}(G) = p - 2$.

Conversely, suppose $\gamma^e_{ct}(G) = p - 2$. Then G cannot contain a cycle of length at least 6, since, if v_1, v_2, v_3, v_4, v_5 and v_6 are consecutive vertices on a cycle C_6, then $V(G) - \{v_1, v_2, v_4, v_5\}$ is an equitable cototal dominating set of G, which is a contradiction.

Now, we have two cases:

Case 1. G containing either $G = C_p; 3 \leq p \leq 5$ containing the consecutive vertices v_1, v_2, v_3 and v_4. If one of the vertices, say v_2, has a neighbor other than v_1 and v_3, then $V(G) - \{v_1, v_2\}$ or $V(G) - \{v_1, v_2, v_3\}$ is an equitable cototal dominating set of $G = C_p$; $3 \leq p \leq 5$ respectively, which is a contradiction. Hence $G = C_p$; $3 \leq p \leq 5$.

Case 2. Let G can be obtained from $C_p; p \leq 5$ by attaching zero or one pendant vertex to at most p vertices of the cycle C_p and G contains a triangle v_1, v_2, v_3, v_1. If each of v_1, v_2 and v_3 has neighbors not on the cycle. Hence $V(G) - \{v_1, N(v_1), N(v_2), N(v_3)\}$ is an equitable cototal dominating set of G, which is a contradiction. Hence v_1 has no neighbors except v_2 and v_3.

If v_2 or v_3 has neighbors not on the cycle, say $N(v_2) = u_2$, then $V(G) - v_1, v_2, u_2$ is an equitable cototal dominating set, a contradiction. Hence every vertex of the cycle is adjacent to degree one vertices not on the cycle.

Proposition 2.5. *For any connected graph G of order p. Then $\gamma_{ct}^{e}(G) = p$ if and only if G is $K_{1,p}; p \geq 3$.*

Proof. Suppose G is $K_{1,p}; p \geq 3$. Thus there exists an equitable cototal dominating set D containing all the vertices of G. Hence the result follows that $\gamma_{ct}^{e}(G) = p$.

Conversely, let G be a connected graph of order p and G is not a star graph $K_{1,p}; p \geq 3$. Then there exists an equitable cototal dominating set of cardinality less than or equal to $p - 1$, a contradiction. Hence G must be $K_{1,p}; p \geq 3$.

Theorem 2.6. *For any connected graph G, $\gamma_{ct}^{e}(G) = 1$ if and only if $\delta(G) \geq 2$ and $\gamma(G) = 1$.*

Proof. Let G be any nontrivial connected graph of order at least 3, then $\gamma_{ct}^{e}(G) = 1$.

We consider two cases:

Case 1. Suppose $\delta(G) = 1$ and $\gamma(G) = 1$. Let $\{u_i\}$ for some i, be the set of vertices of degree $p - 1$ and $\{v_i\}$ be the set of its neighbors. Then by definition of equitable cototal dominating set, $\gamma_{ct}^{e}(G) = |\{u_i\} - \{v_i\}| \geq 2$, a contradiction.

Case 2. Suppose $\delta(G) = 2$. Then there exists a vertex u of maximum degree less than or equal to $p - 1$. Let v be a vertex of minimum degree, then by the definition equitable cototal dominating set $D = \{v_1, v_2, \cdots v_n, v\}$ in G. If $\langle V - D \rangle$ has no isolated vertices, then $|D| \geq 2$. Hence $\gamma_{ct}^{e}(G) = |D| \geq 2$, a contradiction.

Conversely, suppose $\delta(G) \geq 2$ and $\gamma(G) = 1$, then there exists a vertex of degree $p-1$. Therefore $\langle V-D\rangle$ has no isolated vertices. Hence $\gamma_{ct}^{e}(G)=|D|=1$.

Nordhaus - Gaddum provided best possible bound on the sum of the chromatic numbers of a graph and its complement. A corresponding result for the domination number was presented by F. Jaeger et. al [1]. If G is a graph of order $p\geq 2$, then $3\leq \gamma(G)+\gamma(\overline{G})\leq p+1$.

We now give the best possible bounds on the sum and product of the equitable cototal domination number of a graph and its complement.

Theorem 2.7. *Let G be any nontrivial connected graph. If both G and $\overline{G}$ has no isolated vertices, then*

(i) $\gamma_{ct}^{e}(G)+\gamma_{ct}^{e}(\overline{G})\leq p+1$

(ii) $\gamma_{ct}^{e}(G)\cdot\gamma_{ct}^{e}(\overline{G})\leq 2p$.

Finally we conclude this paper by exploring the following open problems.

Open Problem 1. Characterize the graphs for which $\gamma_{t}(G)=\gamma_{t}^{e}(G)$.

Open problem 2. Characterize the graphs for which $\gamma_{t}^{e}(G)=\gamma_{ct}^{e}(G)$.

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