Volume : 3 | Issue : 6 | June 2014 • ISSN No 2277 - 8179

Research Paper

Addition Theorems and Nature of Jacobi Hyperbolic Functions

Mathematics

KEYWORDS : Addition Theorems, Jacobi Elliptic function, Jacobi Hyperbolic function.

Ms. Parveen Bawa Assistant Professor, Deshbandhu College, Delhi University. New Delhi -110019.

ABSTRACT

The discovery of elliptic functions emerged from investigations of integral addition theorems. An addition theorem for a function f is a formula expressing f(u+v) in terms of f(u) and f(v). For a function defined as a definite integral with a variable upper limit, an addition theorem takes the form of an equation between the sum of two such integrals, with upper limits u and v, and an integral whose upper limit is a certain function of u and v.

In this paper a brief, sketch of addition Theorems related to Jacobi hyperbolic functions and the role which the investigation of such addition theorems has played in the development of theory of Jacobi hyperbolic functions is presented. Also from the addition theorems important results depicting the nature of Jacobi hyperbolic functions are discussed.

1. Introduction

An addition theorem for a function f is a formula expressing f(u + v) in terms of f(u) and f(v). In other words, a function f has an addition theorem if there exists a two variable function T such that f(u + v) = T(f(u), f(v)). When the function T is algebraic, we say that f has an algebraic addition theorem. Familiar functions which have algebraic addition theorems are

$f(x) = e^x$, g(x) = sin x, h(x) = tan x, etc.

The corresponding function T for each of the above functions is

T(u, v) = u·v,

$$T(u, v) = uu\sqrt{1-v^2} + v\sqrt{1-u^2},$$

$$T(u, v) = \frac{\tan u + \tan v}{1 - \tan u\ \tan v}$$

respectively (on appropriate intervals).

In [3], Harris Hancock states that "Such an equation offers a means of determining the value of the function for the sum of two quantities as arguments, when the values of the function for the two arguments taken single are known. It is called an algebraic addition theorem."

Addition theorems may apply in different situations. An integral addition theorem is an equation expressing the sum of two integrals with upper limits u and v as a single integral whose upper limit is a function T of u and v . The discovery of elliptic functions emerged from investigations of such integral addition theorems. It was Euler who formulated what is now called the addition theorem for elliptic integrals of the first kind in the middle of the seventeenth century. Euler also obtained the addition theorem for Lemniscate integrals. With Weierstrass [3], the problem of the theory of elliptic functions is to determine all functions of the complex argument for which there exists an algebraic addition-theorem. Later mathematicians, though unable to evaluate this integral in terms of elementary functions, nevertheless obtained useful related results, such as the aforementioned addition theorem due to Euler, as well as a more general addition theorem due to Abel. At around 1860 [4], Weierstrass, impressed by Abel's and Jacobi's work, began his own investigations into the theory of elliptic functions. According to Weierstrass, every function with an algebraic addition theorem is an elliptic function or a limiting case of one (i.e. rational, trigonometric, and exponential functions) [3]. Thus, with Weierstrass's contributions, the intimate connection between algebraic addition theorems and elliptic functions was established. Thus, he obtained:

$$\mathrm{sn}(u+v) = \frac{sn(u)\ cn(v)dn(v) + sn(v)cn(u)dn(u)}{1 - k^2\ sn^2\ (u)sn^2(v)}$$

$$\mathrm{cn}(u+v) = \frac{cn(u)\ cn(v) - sn(v)sn(u)dn(u)dn(v)}{1 - k^2\ sn^2\ (u)sn^2(v)}$$

$$\mathrm{dn}(u+v) = \frac{dn(u)dn(v) - k^2 sn(v)sn(u)dn(u)cn(v)}{1 - k^2\ sn^2\ (u)sn^2(v)}$$

Just as the trigonometric functions have simple algebraic addition theorems, so too, as Abel found, did the elliptic functions.

The aim of the present paper is briefly to present a simple way of describing the properties of Jacobi hyperbolic functions, that at the same time result in new and interesting relationships. In particular this approach leads to the important theorems related to Jacobi hyperbolic functions.

The first part of this paper describes the general theory of Jacobi hyperbolic functions. The second part contains the derivation of addition theorems of Jacobi hyperbolic functions describing the method of evaluation of Jacobi hyperbolic functions when the argument is sum of two quantities. The third part contains the derivation of important theorems depicting the nature of Jacobi hyperbolic functions.

2. Jacobi Hyperbolic functions

[1] With the argument and modulus of hyperbolic functions defined, the functions themselves are just ratios, as in case of trigonometry

shu = sh(u, k) = y
chu = ch(u, k) = x/a
dhu = dh(u, k) = r/a

2.1 Subtraction Theorems

The functions satisfies the two algebraic relations

$ch^2(u, k) - sh^2(u, k) = 1$
$dh^2(u, k) - k^2 sh^2 (u, k) = 1$

2.2 The formulas for differentiating hyperbolic functions.

$$\frac{d}{du} \mathrm{chu} = \mathrm{shu\ dhu}$$

$$\frac{d}{du} \mathrm{shu} = \mathrm{chu\ dhu}$$

$$\frac{d}{du} \mathrm{dhu} = k^2 \mathrm{shu\ chu}$$

3 Addition Theorems of Jacobi Hyperbolic function

Let u +v = α, where u and v vary but α is constant such that $\frac{dv}{du} = -1$

Let S_1 = sh(u) and S_2 = sh(v)

Then $(\dot{S_1})^2 = ch^2 (u).dh^2 (u)$

$= (1 + sh^2 u)\ (1 + k^2 sh^2 u)$

$= (1 + s_1^2) (1 + k^2 s_1^2)$ (i)

Also, $(\dot{S_2})^2 = ch^2 (v).dh^2 (v)$

$= (1 + sh^2 v)\ (1 + k^2 sh^2 v)$

$= (1 + s_2^2) (1 + k^2 s_2^2)$ (ii)

Again differentiating (i) w.r.t u to get

$2 \dot{s_1}\ddot{s_1} = 2 S_1 \dot{s_1} (1 + k^2 s_1^2)+2k^2 S_1 \dot{s_1} (1 + s_1^2)$

$= 2 S_1 \dot{s_1} + 2 k^2 S_1 \dot{s_1} + 4 k^2 S_1^3 \dot{s_1}$

Thus

$\ddot{s_1} = (1 + k^2) s_1 + 2k^2 s_1^3$ (iii)

Again differentiating (ii) w.r.t u to get

$-2 \dot{s_2}\ddot{s_2} = -2 S_2 \dot{s_2} (1 + k^2 s_2^2)-2k^2 S_2 \dot{s_2} (1 + s_2^2)$

$= -2 S_2 \dot{s_2} - 2 k^2 S_2 \dot{s_2} + 4 k^2 S_2^3 \dot{s_2}$

Thus

$\ddot{s_2} = (1 + k^2) s_2 + 2k^2 s_2^3$ (iv)

Now multiply (i) by S_2^2 to get

$(S_2)^2 (\dot{S_1})^2 = (S_2)^2 + (1 + k^2)(S_1)^2 (S_2)^2 + k^2(S_2)^2 S_1^4$ (v)

Now multiply (ii) by S_1^2 to get

$(S_1)^2 (\dot{S_2})^2 = (S_1)^2 + (1 + k^2)(S_1)^2 (S_2)^2 + k^2(S_1)^2 S_2^4$ (vi)

Subtract (v) and (vi) to get

$(S_2)^2 (\dot{S_1})^2 - (S_1)^2 (\dot{S_2})^2 = (S_2^2 - S_1^2)(1 - k^2 s_1^2 s_2^2)$ (vii)

Multiply (iii) by S_2 to get

$\ddot{s_1} s_2 = (1 + k^2) s_1 s_2 + 2k^2 s_1^3 s_2$ (viii)

Multiply (iv) by S_1 to get

$\ddot{s_2} s_1 = (1 + k^2) s_1 s_2 + 2k^2 s_1 s_2^3$ (ix)

Subtract (viii) and (ix) to get

$\ddot{s_1} s_2 - \ddot{s_2} s_1 = 2k^2 s_1 s_2 (s_1^2 - s_2^2)$ (x)

Divide (x) by (vii) to get

$$\frac{\ddot{s_1} s_2 - \ddot{s_2} s_1}{(S_2)^2 (\dot{S_1})^2 - (S_1)^2 (\dot{S_2})^2} = \frac{2k^2 s_1 s_2 (s_1^2 - s_2^2)}{(s_2^2 - s_1^2)(1 - k^2 s_1^2 s_2^2)} \quad \text{(xi)}$$

Multiply both sides of (xi) by $\dot{s_1} s_2 + \dot{s_2} s_1$ to get

$$\frac{\frac{d}{du}(\dot{s_1} s_2 - \dot{s_2} s_1)}{\dot{s_1} s_2 - \dot{s_2} s_1} = \frac{\frac{d}{du}(1 - k^2 s_1^2 s_2^2)}{(1 - k^2 s_1^2 s_2^2)}$$

Integrating yields

$$\frac{\dot{s_1} s_2 - \dot{s_2} s_1}{(1 - k^2 s_1^2 s_2^2)} = C \text{, a constant}$$

but we already know $\dot{s_1}$ *and* $\dot{s_2}$ so

$$\frac{sh(u)\, ch(v) dh(v) + sh(v) ch(u) dh(u)}{1 - k^2\, sh^2(u) sh^2(v)} = C$$

This can be written as a function of two variables f(u + v)

Let v = 0

$$\frac{sh(u)\, ch(0) dh(0) + sh(0) ch(u) dh(u)}{1 - k^2\, sh^2(u) sh^2(0)} = f(u)$$

So f is sh.

Hence sh(u+v) = $\boldsymbol{\frac{sh(u)\, ch(v) dh(v) + sh(v) ch(u) dh(u)}{1 - k^2\, sh^2(u) sh^2(v)}}$

The same method is used to obtain the remaining formula.

4. Nature of Jacobi Hyperbolic Functions

4.1 DEFINITION

A function, f, which is holomorphic except at poles and has singularities in a finite part of the plane and satisfies the following equations:

f(z + ω1) = f(z) and f(z + ω2) = f(z) for some ω1 and ω2 ∈ C is called hyperbolic.

4.2 Jacobi hyperbolic function without poles is constant

Proof. Let f(z) be an Jacobi hyperbolic function with periods ω1 and ω2. f(z) can be enclosed by a parallelogram, Pα, with vertices α, α+ω1, α+ω2 and α+ω1+ω2. As we have no poles and f is bounded, Liouville's Theorem says that f is constant.

4.3 Jacobi hyperbolic functions have order ≥ 2

Proof. If an Jacobi hyperbolic function has order 1, it must have a non-zero residue. However this contradicts lemma 3. So Jacobi hyperbolic functions must have order > 1.

5. Conclusions

The result of above theory indicates the properties of Jacobi hyperbolic functions. They are analogous to trigonometric functions because just like trigonometric functions they also have algebraic addition theorems.

Acknowledgments

I dedicated my work to my husband for his full support.

REFERENCE

1. “Volume 3 Issue 4 2014 ISSN No 2277-8160” Elliptic functions with a view toward Jacobi hyperbolic functions | 2. Lang elliptic Functions Springer-Verlag. | 3. Hancock, H., Lectures on the Theory of Elliptic Integrals analysis, New York: Dover Publications, INC., 1958. | | 4. Kline, Morris, Mathematical Thought from Ancient to Modern Times, vol. 2, Oxford University Press: New York, pages 421, 422, 646, 1990. | | 5. McKean and Moll Elliptic Curves Cambridge University Press (1997). | 6. Synge and Griffith Principles of Mechanics McGraw-Hill. | 7. Roquette Theory of Elliptic Functions. | 8. Whittaker and Watson Course of Modern Analysis Cambridge University Press (1963). |