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# An Overview On Bilevel Programming

Mathematics

KEYWORDS :

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## ABSTRACT

*This paper contains a bibliography of all references central to bilevel programming that the authors know of. To classify some of the references in this review a short overview of some past research in bilevel programming is included. In this bibliography main directions of research as well as main fields of applications and some related methods for solving bilevel programming problems are summarized. We hope this survey facilitates and encourages the research of those who are interested in but unfamiliar with the references in this field.*

**Introduction-**

A large number of mathematical programming problems have an optimization problem in their constraints. Arising from such a situation is a bilevel programming problem. These problems differ from ordinary optimization problems, as it has two levels of optimization tasks, the upper level or outer optimization task and the lower level or inner optimization task. The lower level optimization task appears as a constraint to the upper level optimization task, such that only a global optimal solution to the lower level problem may be a feasible candidate to the upper level optimization problem. This caveat makes bilevel optimization a challenging task, which demands immense computational resources to successfully solve even smaller instances of the problem.

**Motivation-**

A hierarchical structure appears naturally in many occurrences of decision making, and so bilevel programmes can be applied to many areas of optimisation. Thes are commonly found in many practical problems appearing in transportation (network design, optimal pricing), economics (Stackelberg games, principal-agent problem, taxation, policy decisions), management (network facility location, coordination of multi-divisional firms), engineering (optimal design, optimal chemical equilibria) etc.

**Literature Review-**

In terms of modeling, bilevel problems are programs which have a subset of their variables (lower level variable) constrained to be an optimal solution of another problem (lower level problem) parameterized by the remaining variables (upper level variable). A bilevel programming problem in its general form need not be well-defined if the parameterized lower level problem has a nonunique (global) optimal solution for at least one value of the parameter x (the upper level variable). Then, at least two ways out of this unpleasing situation can be found in the literature: First the so-called optimistic or weak formulation and, second, the pessimistic or strong formulation. For an introduction to the optimistic and pessimistic formulation one may refer a book by Dempe [10]. In most papers the optimistic formulation is used and all reformulations of the bilevel problem as optimization problem with equilibrium or variational inequality constraints are possible only in that case.

Applications have been a stimulating factor for the development of bilevel programming. The number of papers presenting applications is growing rapidly. An overview of applications of bilevel programming is given by Marcotte and Savard [22]. Applications in economics include the investigation of networks of oligopolies in [19], hierarchical structures of multidivisional organizations and many more. Principal agent theory has been considered in a number of papers, including [26]. Interesting applications are the determination of optimal prices, as road tolls or prices for electricity discussed by Labbe, Marcotte and Savard in [18] and by Hobbs and Nelson in [13] respectively. Related are the determination of optimal tax credits for bio-fuel production. Agricultural planning problems, facility location problem and defense problems have been investigated. A second large field of applications is in ecological problems; the question of subsidy options to reduce greenhouse-gas emissions or pollution control policies, the disposal of hazardous waste [17] are included in this area.

When designing a subnetwork to be used by carriers of hazardous material, it makes sense to take into account the beheviour of carriers, who may favour shortest routes over safest routes. A bilevel formulation corresponding to this situation is analyzed in [17]. Bilevel programming subsumes the principal-agent paradigm, a classical problem of economics, whereby the leader (principal) sub contracts a job to an agent (follower). The agent is rewarded by the principal according to the quality of some random outcome that determined the leader's revenue. At the lower level, the agent maximizes an objective that is a function of its reward and its e ort level. Of course, the larger the e ort level, the larger the expected revenue associated with the outcome. Whenever the set of outcome is continuous, the resulting lower level problem is in nite-dimensional. See [29] for an overview of the topic. The energy sector, in particular the power sector, has been the topic of some interesting bilevel modelizations. Hobbs and Nelson [13] considered an electric utility that "seeks to minimize costs or maximize bene ts while controlling electric rates and subsidising energy conservation programs". The BPP can easily be interpreted in terms of Stackelberg games which are a special case of them widely used in economics. In stakelberg games, basically there are two decision makers which select their actions in a hierarchical manner. For a detailed discussion one may refer [10]. Bilevel programming problems are more general than stackelberg games in the sense that both admissible stratergy sets can also depend on the choice of the other decision maker. Revenue management is a generic term that covers a set of optimization procedures aimed at maximizing the pro tability of rms characterized by high investment costs, low operating costs, and perishable inventories. It was implemented in the airline industry under the name "yield management", and involved four issues namely ticket pricing, seat allocation, demand forecasting and overbooking. Such model, that involves the pricing and seat allocation policies is described in [23].

The existence of a vertex optimal solution has been verified for the linear problem, for quasiconcave bilevel optimization problems, for problems with quadratic lower level problem, for problems with fractional lower level problems and for problems with bottleneck objective functions. For bilevel programming problems with non-unique lower level optimal solution it is important to investigate the question of how to define a solution. This is discussed by Dempe [12]. Also the significance of the order of the play has been investigated by Bard and Falk [4].

Optimality conditions are one of the central topics in optimization theory. For bilevel programming problems, different approaches have been given in [10]. To formulate optimality conditions it is often necessary to use a one-level reformulation of the bilevel programming problem. A first attempt in this direction by replacing the lower level problem with an infinite number of constraints can be found in the paper [2] given by Bard. A second approach to formulate necessary and sufficient optimality conditions uses assumptions guaranteeing that the lower level problem has a unique strongly stable optimal solution [11]. In this case all the results known from nondifferentiable optimization can be used. Necessary optimality conditions using a reformulation of the bilevel problem by the help of the optimal value function of the lower level problem can be found in [29]. Malhotra and Arora [21] have applied duality theory to the lower level problem which is used to derive a minimax problem and to develop optimality conditions. Necessary optimality conditions of Karush-Kuhn-Tucker type can be found in the paper [8] by Chen and Florian. Another attempt deals with the use of a penalty function approach [30]. The special case of a quadratic lower level problem is addressed in [27].

The comprehensive insight into methods globally solving bilevel programming problems is given by Bard [3]. Historically the first methods aimed to solve the problem globally. The monograph [3] describes a large number of such algorithms. For linear bilevel programming problems the fact that an optimal solution can be found at a vertex of the underlying poly-hedron can be used. This results in vertex enumeration methods and in the complementary pivoting algorithm for solving linear bilevel problems and for problems with quadratic lower level problems ([7], [15], [16]). In the K-th best algorithm the computation of non-optimal vertex solutions in the lower level problem is used to solve the bilevel problem globally by Bialas and Karwan [6]. Branch-and-bound methods where the complementarity constraints of the Karush-Kuhn-Tucker reformulation of the lower level problem have been relaxed can be found in [4]. The approach was adapted to linear-quadratic programs by Bard and Moore [5] in 1990 and to the quadratic case by Al-Khayal, Horst and Pardalos [1] in 1992. Algorithms based on bicriteria optimization were developed but were shown to be inadequate later. The linear bilevel programming problem has been attacked by transforming the Karush-Kuhn-Tucker reformulation into mixed-discrete linear constraints and solving the resulting problem in [28].

Bilevel programming problems are nonconvex optimization problems. Due to the inherent difficulties for solving such problems globally, many researchers focus their investigations on deriving descent methods for computing stationary solutions. For the linear bilevel programming problem this enables one to compute a local optimal solution as given discussed by Dempe [9]. An interior point algorithm solving the Karush-Kuhn-Tucker reformulation of the bilevel programming problem is given in [20] by Leontiev and Herskovits, penalty functions are used in [23], [25] whereas a double penalty method was proposed by Ishizuka and Aiyoshi [14].

Scope and Future Work- The bilevel problems are closely related to many other optimization problems which is often a wise approach to solve this problem. Many theoretical results can be established connecting bilevel problems to other optimization problems in mathematical programming thereby developing a new way to deal with such problems. Bilevel problems being intrinsically difficult, it is not surprising that most algorithmic research to date has focused on the simplest cases of bilevel programs possessing nice properties such as linear, quadratic or convex objective and/or constraint functions under the optimistic approach. Hence more complex bilevel programs including the pessimistic case builds a strong scope for research in this area. Some approximate methodologies can be used to replace the lower level problem and convert the whole problem into a single-level optimization task.

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