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On Bounds For Weighted Fuzzy Mean Difference-Divergence Measures

Mathematics

KEYWORDS : Weighted Fuzzy mean of order t, Weighted Fuzzy mixed means, N_1,N_2 and N_3, Weighted fuzzy Geometric, Harmonic,Mixed means difference-divergences.

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ABSTRACT

In the present communication, we have considered the bounds and inequalities among weighted fuzzy mean difference-divergence measures. Here we have considered mixed weighted fuzzy mean difference-divergences, their bounds and inequalities such as :

(i) $M_{N_3N_1}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_3H}(\mu_A(x_i),\mu_B(x_i);W)$

(ii) $M_{N_2N_3}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_2N_1}(\mu_A(x_i),\mu_B(x_i);W) \le \frac{1}{3} M_{N_2G}(\mu_A(x_i),\mu_B(x_i);W)$

$\le M_{N_2H}(\mu_A(x_i),\mu_B(x_i);W)$ *and*

(iii) $M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_1H}(\mu_A(x_i),\mu_B(x_i);W)$

To establish the bounds/inequalities, we have exploited weighted fuzzy f-divergence corresponding to Csiszar's f-divergence.

1.1 INTRODUCTION :

It is a well known fact that means-Arithmetic, Geometric and harmonic have a vital role for application in mathematical sciences, statistical analysis medical sciences, budget analysis, planning, environmental sciences and many others. Since the origin of fuzzy set theoretic measures of information, there is a wide application of fuzzy measures of information-divergences in diverse field of cybernetics, engineering, management, medicine, data base, fuzzy numbers and operators, diagnosis and environmental modeling, classification and recognition, scheduling and planning, cognition, multi-media, signal processing, interpersonal communication, air craft control, Expert system and many other areas (C.f G.J. Klir and T.A. Folger) [3]. Importance of an event or experiment, has been the outlook of every human being, therefore, we will utilize the weighted distribution corresponding to fuzzy set theoretic distribution and consider the following fuzzy information scheme for the purpose :

$$F.S_1 = \begin{bmatrix} E_1 & E_2 \ldots\ldots\ldots\ldots\ldots E_n \\ \mu_A(x_1) & \mu_A(x_2) \ldots\ldots\mu_A(x_n) \\ w_1 & w_2 \ldots\ldots\ldots\ldots\ldots w_n \end{bmatrix} \quad \ldots..(1.1)$$

For the scheme (1.1) Kapur [2] defined the weighted Fuzzy divergence entropy as :

$$F[(\mu_A(x_i) ; W_i] = -\sum_{i=1}^{n} w_i \, [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))] \quad \ldots\ldots(1.2)$$

For the study of weighted fuzzy divergence measure, we define the weighted information scheme as follows:

$$F.S_2 = \begin{bmatrix} E_1 & E_2 \ldots\ldots\ldots\ldots\ldots E_n \\ \mu_A(x_1) & \mu_A(x_2) \ldots\ldots\mu_A(x_n) \\ \mu_B(x_1) & \mu_B(x_2) \ldots\ldots\mu_B(x_n) \\ w_1 & w_2 \ldots\ldots\ldots\ldots\ldots w_n \end{bmatrix} \quad (1.3)$$

and the weighted fuzzy divergence measure has been defined by kapur [2]

$$D(\mu_A(x_i), \mu_B(x_i) : W) = \sum_{i=1}^{n} w_i \left[\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) \log \frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))} \right] \ldots..(1.4)$$

Since the study of divergence measures either probabilistic or fuzzy, is important due to reliability of the system probabilistic or fuzzy, so analogous to probabilistic divergences, Singh and Tomar [8] have defined and studied a series of symmetric and non-symmetric fuzzy divergence measures. Taneja [9], Zhi-Hua Zhang and Yu-Dong Wu [10], H. N. Shi, J. Zhang and Da Mao Li [2] and Huan Nanshi and Jian Zhang [3], Ladislav Makjicka [6] have studied many types of means and characterized them and established bounds for them. Earlier Singh and Tomar [8] have studied fuzzy mean difference-divergences also. Recently Priti, Santosh and Singh [7] have studied some inequalities among them. In this communication, we enhance studies further for weighted mean difference-divergences, first defining weighted fuzzy mean of order t, t ≠ 0 and for different values of t, i.e. t = –1, t = 0, t = 1, t = ½, t = 2, and t = ± ∞, particular cases have been defined. Some mixed weighted mean difference-divergence measures also have been defined. Section 2 presents the weighted fuzzy mean difference-divergences which are needed for the concerned bounds, their functional forms, first and second derivatives. Section 3, presents some basic weighted fuzzy divergences analogous to Csiszars' [1] f-divergence and the bounds in the form of inequalities which have been established through prepositions exploiting calculus. Now let us define weighted fuzzy mean of order t, t ≠ 0 as follows:

$$M_t(\mu_A(x_i), \mu_B(x_i); W) =$$

$$\begin{cases} w_i\left\{\left(\dfrac{\mu_A^t(x_i)+\mu_B^t(x_i)}{2}\right)^{\frac{1}{t}} + \left(\dfrac{(1-\mu_A(x_i))^t + (1-\mu_B(x_i))^t}{2}\right)^{\frac{1}{t}}\right\}, & \text{when } t \neq 0 \\[2ex] \text{Weighted Fuzzy Harmonic Mean} = H(\mu_A(x_i), \mu_B(x_i); W) \\ \quad = w_i\left[\dfrac{2\mu_A(x_i)\,\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \dfrac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right], & \text{when } t=-1 \\[2ex] \text{Weighted Fuzzy Geometric Mean} = G(\mu_A(x_i), \mu_B(x_i); W) \\ \quad = w_i\sqrt{\mu_A(x_i)\,\mu_B(x_i) + (1-\mu_A(x_i))(1-\mu_B(x_i))}, & \text{when } t=0 \\[2ex] \text{Weighted Fuzzy Arithmetic Mean} = A(\mu_A(x_i), \mu_B(x_i); W) \\ \quad = w_i\left\{\dfrac{\mu_A(x_i)+\mu_B(x_i)}{2} + \dfrac{2-\mu_A(x_i)-\mu_B(x_i)}{2}\right\}, & \text{when } t=1 \\[2ex] \text{Weighted Fuzzy Root Mean Square} = S(\mu_A(x_i), \mu_B(x_i); W) \\ \quad = w_i\left\{\sqrt{\dfrac{(\mu_A(x_i))^2 + (\mu_B(x_i))^2}{2}} + \sqrt{\dfrac{(1-\mu_A(x_i))^2 + (1-\mu_B(x_i))^2}{2}}\right\}, & \text{when } t=2 \\[2ex] \text{Weighted Fuzzy Mixed Mean} = N_1(\mu_A(x_i), \mu_B(x_i); W) \\ \quad = w_i\left[\left(\dfrac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)^2 + \left(\dfrac{\sqrt{(1-\mu_A(x_i))}+\sqrt{(1-\mu_B(x_i))}}{2}\right)^2\right], & \text{when } t=\dfrac{1}{2} \\[2ex] \textit{Weighted}\max\{\mu_A(x_i), \mu_B(x_i); W\}, & \text{when } t=\infty \\ \textit{Weighted}\min\{\mu_A(x_i), \mu_B(x_i); W\}, & \text{when } t=-\infty \end{cases}$$

MIXED WEIGHTED FUZZY MEAN MEASURES

1. $N_2\left(\mu_A(x_i), \mu_B(x_i); W\right) = \sqrt{N_1\left(\mu_A(x_i), \mu_B(x_i); W\right) A\left(\mu_A(x_i), \mu_B(x_i); W\right)}$

$$= w_i\left[\left\{\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right\}^2 + \left\{\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\right\}^2 \times \left\{\frac{\mu_A(x_i)+\mu_B(x_i)}{2} + \frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}\right\}\right]$$

or

$$= w_i\left[\left\{\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right\}\left\{\sqrt{\frac{\mu_A(x_i)+\mu_B(x_i)}{2}}\right\}3 + \left\{\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\right\}\left\{\sqrt{\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}}\right\}\right.$$

2. $N_3\left(\mu_A(x_i), \mu_B(x_i); W\right) = \dfrac{2A\left(\mu_A(x_i), \mu_B(x_i); W\right)+G\left(\mu_A(x_i), \mu_B(x_i); W\right)}{3}$

$$= w_i\left[\frac{\left(\mu_A(x_i)+\mu_B(x_i)\right)+\left(1-\mu_A(x_i)\right)+\left(1-\mu_B(x_i)\right) + \sqrt{[\mu_A(x_i)\,\mu_B(x_i)+(1-\mu_A(x_i)1-\mu_B(x_i))]}}{3}\right]$$

or

$$= w_i\left[\left\{\frac{\mu_A(x_i)+\mu_B(x_i)+\sqrt{\mu_A(x_i)\,\mu_B(x_i)}}{3}\right\} + \left\{\frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))\,(1-\mu_B(x_i))}}{3}\right\}\right]$$

Now we define the mean differences which are necessary for this study, in the next section.

SECTION-2

2.1 WEIGHTED FUZZY MEAN DIFFERENCE-DIVERGENCE MEASURES

Let us consider the following mean difference-divergence measures

1. $M_{N_2N_3}(A//B;W)$

$$= \sum_{i=1}^{n} w_i\left\{\left(\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)\left(\sqrt{\frac{\mu_A(x_i)+\mu_B(x_i)}{2}}\right) + \left(\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\right)\left(\sqrt{\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}}\right) - \left(\frac{\mu_A(x_i)+\mu_B(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}}{3}\right) - \left(\frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}\right)\right\}$$

2. $M_{N_2N_1}(A//B;W)$

$$= \sum_{i=1}^{n} w_i\left\{\left(\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)\left(\sqrt{\frac{\mu_A(x_i)+\mu_B(x_i)}{2}}\right) + \left(\frac{\sqrt{(1-\mu_A(x_i))}+\sqrt{(1-\mu_B(x_i))}}{2}\right)\left(\sqrt{\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}}\right) - \left(\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)^2 - \left(\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\right)^2\right\}$$

3. $M_{N_2G}(A//B;W)$

$$= \sum_{i=1}^{n} w_i\left\{\left(\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)\left(\sqrt{\frac{\mu_A(x_i)+\mu_B(x_i)}{2}}\right) + \left(\frac{\sqrt{(1-\mu_A(x_i))}+\sqrt{(1-\mu_B(x_i))}}{2}\right)\times\left(\sqrt{\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}}\right) - \left(\sqrt{\mu_A(x_i)\mu_B(x_i)+\left(1-\mu_A(x_i)\right)\left(1-\mu_B(x_i)\right)}\right)\right\}$$

4. $M_{N_2H}(A//B;W) =$

$$\sum_{i=1}^{n} w_i \left\{ \left(\frac{\sqrt{\mu_A(x_i)} + \sqrt{\mu_B(x_i)}}{2} \right) \left(\sqrt{\frac{\mu_A(x_i) + \mu_B(x_i)}{2}} \right) \right.$$

$$+ \left(\frac{\sqrt{(1-\mu_A(x_i))} + \sqrt{1-\mu_B(x_i)}}{2} \right) \left(\sqrt{\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}} \right)$$

$$\left. - \left(\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)} \right) - \left(\frac{2\,1(-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)} \right) \right\}$$

5. $M_{N_3N_1}(A//B;W) =$

$$\sum_{i=1}^{n} w_i \left\{ \frac{\mu_A(x_i) + \mu_B(x_i) + \sqrt{\mu_A(x_i)\mu_B(x_i)}}{3} \right.$$

$$+ \frac{2-\mu_A(x_i)-\mu_B(x_i) + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}$$

$$- \left(\frac{\sqrt{\mu_A(x_i)} + \sqrt{\mu_B(x_i)}}{2} \right)^2$$

$$\left. - \left(\frac{\sqrt{(1-\mu_A(x_i))} + \sqrt{(1-\mu_B(x_i))}}{2} \right)^2 \right\}$$

6. $M_{N_3G}(A//B;W) =$

$$\sum_{i=1}^{n} w_i \left\{ \left(\frac{\mu_A(x_i) + \mu_B(x_i) + \sqrt{\mu_A(x_i)\mu_B(x_i)}}{3} \right) \right.$$

$$+ \left(\frac{2-\mu_A(x_i)-\mu_B(x_i) + \sqrt{1-\mu_A(x_i))\,(1-\mu_B(x_i))}}{3} \right)$$

$$\left. - \left(\sqrt{\mu_A(x_i)\mu_B(x_i) + (1-\mu_A(x_i))\,(1-\mu_B(x_i))} \right) \right\}$$

7. $M_{N_3H}(A//B;W) =$

$$\sum_{i=1}^{n} w_i \left\{ \left(\frac{\mu_A(x_i) + \mu_B(x_i) + \sqrt{\mu_A(x_i)\mu_B(x_i)}}{3} \right) \right.$$

$$+ \left(\frac{2-\mu_A(x_i)-\mu_B(x_i) + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3} \right)$$

$$\left. - \left(\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)} \right) - \left(\frac{2\,1(-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)} \right) \right\}$$

8. $M_{N_1G}(A//B;W) =$

$$\sum_{i=1}^{n} w_i \left\{ \left(\frac{\mu_A(x_i) + \mu_B(x_i) + 2\sqrt{\mu_A(x_i)\,\mu_B(x_i)}}{4} \right) \right.$$

$$+ \left(\frac{2-\mu_A(x_i)-\mu_B(x_i) + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{4} \right)$$

$$\left. - \left(\sqrt{2\mu_A(x_i)\mu_B(x_i) + 1 - \mu_A(x_i) - \mu_B(x_i)} \right) \right\}$$

9. $M_{N_1H}(A//B;W) =$

$$\sum_{i=1}^{n} w_i \left\{ \left(\frac{\mu_A(x_i) + \mu_B(x_i) + 2\sqrt{\mu_A(x_i)\mu_B(x_i)}}{4} \right) \right.$$

$$+ \left(\frac{2-\mu_A(x_i)-\mu_B(x_i) + 2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{4} \right)$$

$$\left. - \left(\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)} \right) - \left(\frac{2\,1(-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)} \right) \right\}$$

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FUNCTIONAL FORM, FIRST AND SECOND DERIVATIVES

MEAN DIFFERENCE-DIVERGENCE	FUNCTIONAL FORM	FIRST DERIVATIVE	SECOND DERIVATIVE
$M_{N_2N_3}(A//B)$	$f_{N_2N_3}(x) = \left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right) + \left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{\frac{3}{2}-x}{2}}\right) - \left(\frac{x+\frac{1}{2}+\sqrt{\frac{x}{2}}}{3}\right) - \left(\frac{\frac{3}{2}-x+\sqrt{\frac{1-x}{2}}}{3}\right)$	$f'_{N_2N_3}(x) = \frac{1}{4\sqrt{x}}\left(\frac{x+\frac{1}{2}}{2}\right)^{\frac{1}{2}} + \frac{1}{8}\left(\sqrt{x}+\sqrt{\frac{1}{2}}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{\frac{1}{2}} - \frac{1}{4\sqrt{1-x}}\left(\frac{\frac{3}{2}-x}{2}\right)^{\frac{1}{2}} - \frac{1}{8}\left(\sqrt{1-x}+\sqrt{\frac{1}{2}}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}} - \frac{1}{6\sqrt{2}\sqrt{x}} + \frac{1}{6\sqrt{2}(\sqrt{1-x})}$	$f''_{N_2N_3}(x) = \frac{1}{8}x^{-\frac{1}{2}}\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}\left(\frac{x+\frac{1}{2}}{2}\right)^{\frac{1}{2}} - \frac{1}{32}\left\{\left(\sqrt{x}+\sqrt{\frac{1}{2}}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{3}{2}}\right\} - \frac{1}{8}(1-x)^{-\frac{3}{2}}\left(\frac{\frac{3}{2}-x}{2}\right)^{\frac{1}{2}} - \frac{1}{32}\left(\sqrt{1-x}+\sqrt{\frac{1}{2}}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{3}{2}} + \frac{1}{12\sqrt{2}}x^{-\frac{3}{2}} + \frac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}}$
$M_{N_2N_1}(A//B)$	$f_{N_2N_1}(x) = \left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right) + \left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)$	$f'_{N_2N_1}(x) = \frac{1}{4\sqrt{2}}\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{\sqrt{x+\frac{1}{2}}}\right) + \frac{1}{4\sqrt{2}}\frac{\sqrt{x+\frac{1}{2}}}{\sqrt{x}} + \frac{1}{4\sqrt{2}}$	$f''_{N_2N_1}(x) = \frac{1}{2}\left(x^2+\frac{1}{4}\right)^{-\frac{1}{2}} - \frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}} + \frac{1}{2}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{\frac{1}{2}}$

$M_{N_2N_1}(A//B)$	$\left(\sqrt{\frac{\frac{3}{2}-x}{2}}\right)$ $-\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)^2$ $-\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)^2$	$\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}-x}}\right)$ $+\frac{1}{4\sqrt{2}}\frac{\sqrt{\frac{3}{2}-x}}{\sqrt{1-x}}$ $-\frac{1}{4\sqrt{2}}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{1-x}}\right)$	$-\left(\frac{1-x}{2}\right)^2\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{3}{2}}$ $+\frac{1}{4(2x)^{\frac{3}{2}}}+\frac{1}{4(2(1-x))^{\frac{3}{2}}}$
$M_{N_2G}(A//B)$	$f_{N_2G}(x)=$ $\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right)$ $+\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)$ $\left(\sqrt{\frac{\frac{3}{2}-x}{2}}\right)$ $-\left(\sqrt{\frac{x}{2}+\frac{1-x}{2}}\right)$	$f'_{N_2G}(x)=$ $\frac{1}{4\sqrt{2}}\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{\sqrt{x+\frac{1}{2}}}\right)$ $+\frac{1}{4\sqrt{2}}\frac{\sqrt{x+\frac{1}{2}}}{\sqrt{x}}$ $-\frac{1}{4\sqrt{2}}\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}-x}}$ $-\frac{1}{4\sqrt{2}}\frac{\sqrt{\frac{3}{2}-x}}{\sqrt{1-x}}$	$f''_{N_2G}(x)=$ $\frac{1}{2}\left(x^2+\frac{1}{4}\right)^{-\frac{1}{2}}$ $-\frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}$ $+\frac{1}{2}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{\frac{1}{2}}$ $-\left(\frac{1-x}{2}\right)^2\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{3}{2}}$ $+\frac{1}{16}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{16}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}$
$M_{N_2H}(A//B)$	$f_{N_2H}(x)=$ $\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right)$ $+\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)$	$f'_{N_2H}(x)=$ $\frac{1}{4\sqrt{2}}\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{\sqrt{x+\frac{1}{2}}}\right)$	$f''_{N_2H}(x)=$ $-\frac{1}{8\sqrt{2}}\left[\left(\sqrt{x}+\sqrt{\frac{1}{2}}\right)\right.$ $\left(x+\frac{1}{2}\right)^{-\frac{3}{2}}-\left(x+\frac{1}{2}\right)^{-\frac{1}{2}}$

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	$\left(\sqrt{\dfrac{\frac{3}{2}-x}{2}}\right)$ $-\left(\dfrac{x}{x+\frac{1}{2}}-\dfrac{1-x}{\frac{3}{2}-x}\right)$	$\left.+\dfrac{\sqrt{x+\frac{1}{2}}}{\sqrt{x}}\right)-\dfrac{1}{4\sqrt{2}}$ $\left(\dfrac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}-x}}\right)$ $-\dfrac{1}{4\sqrt{2}}\dfrac{\sqrt{\frac{3}{2}-x}}{\sqrt{1-x}}$ $-\dfrac{1}{2\left(x+\frac{1}{2}\right)^{2}}$ $+\dfrac{1}{2\left(\frac{3}{2}-x\right)^{2}}$	$\left.x^{-\frac{1}{2}}\right]-\dfrac{1}{8\sqrt{2}}\left[\left(x+\dfrac{1}{2}\right)^{\frac{1}{2}}\right.$ $\left.x^{-\frac{3}{2}}-x^{-\frac{1}{2}}\left(x+\dfrac{1}{2}\right)^{-\frac{1}{2}}\right]$ $-\dfrac{1}{8\sqrt{2}}\left[\left(\sqrt{1-x}+\sqrt{\dfrac{1}{2}}\right)\right.$ $\left(\dfrac{3}{2}-x\right)^{-\frac{3}{2}}+\left(\dfrac{3}{2}-x\right)^{-\frac{1}{2}}$ $\left.(1-x)^{-\frac{1}{2}}\right]-\dfrac{1}{8\sqrt{2}}$ $\left[\left(\dfrac{3}{2}-x\right)^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}\right.$ $\left.-(1-x)^{-\frac{1}{2}}\left(\dfrac{3}{2}-x\right)^{-\frac{1}{2}}\right]$ $-\dfrac{1}{2\left(x+\frac{1}{2}\right)^{2}}+\dfrac{1}{2\left(\frac{3}{2}-x\right)^{2}}$
$M_{N_3N_1}(A//B)$	$f_{N_3N_1}(x)=$ $\left(\dfrac{x+\frac{1}{2}+\sqrt{\frac{x}{2}}}{3}\right)+$ $+\left(\dfrac{\frac{3}{2}-x+\sqrt{\frac{1-x}{2}}}{3}\right)$ $-\left(\dfrac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)^{2}$ $-\left(\dfrac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)^{2}$	$f'_{N_3N_1}(x)=$ $\dfrac{1}{6\sqrt{2}\sqrt{x}}-$ $-\dfrac{1}{6\sqrt{2\sqrt{1-x}}}$ $-\dfrac{1}{4\sqrt{2}\sqrt{x}}$ $+\dfrac{1}{4\sqrt{2\sqrt{1-x}}}$	$f''_{N_3N_1}(x)=$ $-\dfrac{1}{12\sqrt{2}}x^{-\frac{3}{2}}-$ $-\dfrac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}}$ $+\dfrac{1}{8\sqrt{2}}x^{-\frac{3}{2}}$ $-\dfrac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}}$

$M_{N_3G}(A//B)$	$f_{N_3G}(x) = \left(\frac{x+\frac{1}{2}+\sqrt{\frac{x}{2}}}{3}\right) + \left(\frac{\frac{3}{2}-x+\sqrt{\frac{1-x}{2}}}{3}\right) - \left(\sqrt{\frac{x}{2}+\frac{1-x}{2}}\right)$	$f'_{N_3G}(x) = \frac{1}{3}\left(\frac{1}{2\sqrt{2}\sqrt{x}} - \frac{1}{2\sqrt{2}\sqrt{1-x}}\right)$	$f''_{N_3G}(x) = -\frac{1}{12\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}}$
$M_{N_3H}(A//B)$	$f_{N_3H}(x) = \left(\frac{x+\frac{1}{2}+\sqrt{\frac{x}{2}}}{3}\right) + \left(\frac{\frac{3}{2}-x+\sqrt{\frac{1-x}{2}}}{3}\right) - - \left(\frac{x}{x+\frac{1}{2}} - \frac{1-x}{\frac{3}{2}-x}\right)$	$f'_{N_3H}(x) = \frac{1}{3}\left(\frac{1}{2\sqrt{2}\sqrt{x}} - \frac{1}{2\sqrt{2}\sqrt{1-x}}\right) - \frac{1}{2\left(x+\frac{1}{2}\right)^2} + \frac{1}{2\left(\frac{3}{2}-x\right)^2}$	$f''_{N_3H}(x) = -\frac{1}{12\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}} + \left(x+\frac{1}{2}\right)^{-3} + \left(\frac{3}{2}-x\right)^{-3}$
$M_{N_1G}(A//B)$	$f_{N_1G}(x) = \left(\frac{x+\frac{1}{2}+\sqrt{2x}}{4}\right) + \left(\frac{\frac{3}{2}-x+\sqrt{2(1-x)}}{4}\right) - \left(\sqrt{x+\left(\frac{1}{2}-x\right)}\right)$	$f'_{N_1G}(x) = \frac{1}{4}\left(\frac{1}{\sqrt{2}\sqrt{x}} - \frac{1}{\sqrt{2}\sqrt{1-x}}\right)$	$f''_{N_1G}(x) = -\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}}$

$M_{N_1H}(A//B)$	$f_{N_1H}(x)=$ $\left(\frac{x+\frac{1}{2}+\sqrt{2x}}{4}\right)$ $+\left(\frac{\frac{3}{2}-x+\sqrt{2(1-x)}}{4}\right)$ $-\left(\frac{x}{x+\frac{1}{2}}\right)-\left(\frac{1-x}{\frac{3}{2}-x}\right)$	$f'_{N_1H}(x)=$ $\frac{1}{4}\left(\frac{1}{\sqrt{2x}}-\frac{1}{\sqrt{2(1-x)}}\right)$ $-\frac{1}{2\left(x+\frac{1}{2}\right)^2}$ $+\frac{1}{2\left(\frac{3}{2}-x\right)^2}$	$f''_{N_1H}(x)=$ $-\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}}-\frac{1}{8\sqrt{2}}$ $(1-x)^{-\frac{3}{2}}+\left(x+\frac{1}{2}\right)^{-3}$ $+\left(\frac{3}{2}-x\right)^{-3}$

SECTION-3

3.1 BOUNDS FOR WEIGHTED FUZZY MIXED MEAN DIFFERENCE- DIVERGENCES

As the main objective of this communication is to consider the bounds or inequalities among Weighted Fuzzy Mixed Mean Difference-Divergence measures considered in section 2, so we consider their bounds. As we have mentioned earlier that we exploit Csiszar's [1] f-divergence extended for fuzzy weighted divergence measures, so we define the weighted fuzzy f-divergence measure as follows :

WEIGHTED FUZZY f-DIVERGENCE MEASURE :

Csiszar's [1] f-divergence is defined as :

Def 1 : Given a convex function $f:[0,\infty)\to R$, the f-divergence measure is given by :

$$C_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \quad \text{……(3.1)}$$

where $P,Q\in R_+$.

Def 2 : Csiszar's f-divergence for weighted distribution,

$$W = (w_1, w_2, ..., w_n),\ w_i > 0$$

Let $f:[0,\infty)\to R$ and $w_i>0$, for all $i = 1,2,........,n$ then

$$C_f(P,Q\,;W) = \sum_{i=1}^{n} w_i\, q_i f\left(\frac{p_i}{q_i}\right) \quad \text{……(3.2)}$$

so weighted fuzzy f-divergence analogous to probability distribution can be defined as :

$$C_f(\mu_A(x_i),\mu_B(x_i)\,;W) = \sum_{i=1}^{n} w_i\,\mu_B(x_i) f\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) \quad \text{…(3.3)}$$

where $A=\{\mu_A(x_i):\forall i=1,2,......,n\}$

$B=\{\mu_B(x_i):\forall i=1,2,......,n\}$

Hence (3.3) is called the Weighted Fuzzy f-Divergence Measure.

Now in this section, we will exploit (3.3) for mixed weighted mean difference divergence measures.

THEOREM 1 : Let $f_1, f_2 : I\subset R\to R$ be twice differentiable convex functions which are normalized i.e. $f_1(1) = f_2(1) = 0$ and suppose that

(i) f_1 and f_2 are twice differentiable on $(a,b)\subset I$

(ii) There exist real constants m and M such that $m<M$ and

$$m \le \frac{f_1''(x)}{f_2''(x)} \le M \quad \text{……(3.4)}$$

where

$f_2''(x)>0,\ \forall x\in(a,b)\subset I$ (c.f. Taneja [4])

Then we have

$$mC_{f_2}(P//Q) \le C_{f_1}(P//Q) \le MC_{f_2}(P//Q) \quad \text{……(3.5)}$$

Extension for Weighted Fuzzy Distribution :

Using (3.3), we extend (3.5) as

$$mC_{f_2}(A//B;W) \le C_{f_1}(A//B;W) \le MC_{f_2}(A//B;W) \quad \text{……(3.5a)}$$

Now we are able to consider the boudns for Weighted Fuzzy Mean Difference-Divergences which we consider in the next theorem.

THEOREM 2 :

$$M_{N_3N_1}(\mu_A(x_i),\mu_B(x_i)\,;W) \le M_{N_1G}(\mu_A(x_i),\mu_B(x_i)\,;W) \le M_{N_3H}(\mu_A(x_i),\mu_B(x_i)\,;W) \quad \text{…..(3.6)}$$

To prove the theorem 2, we have the following propositions:

PROPOSITION 1 : For lower bound of $M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W)$, we have

$$M_{N_3N_1}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_1G}(\mu_A(x_i),\mu_B(x_i)\,;W) \quad \text{…….(3.7)}$$

PROOF : Let us define :

$$g_{N_3N_1-N_1G}(x) = \frac{f''_{N_3N_1}(x)}{f''_{N_1(G)}(x)} \quad ....(3.8)$$

where $f''_{N_3N_1}(x)$ and $f''_{N_1G}(x)$ are taken from functional forms.

$g_{N_3N_1-N_1G}(x) =$

$$\frac{-\frac{1}{12\sqrt{2}}x^{\frac{-3}{2}}-\frac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}}+\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}}-\frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}}}{-\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}}-\frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}}}$$

$$=\frac{p}{Q} \quad ......(3.9)$$

Differentiating (3.9) w.r.t. x, we get

$$g'_{N_3N_1-N_1G}(x) = \frac{P'Q-PQ'}{Q^2} \quad ......(3.10)$$

From (3.10), we observe that

$$g'_{N_3N_1-N_1G}(x)\begin{cases}\geq 0 \ , \text{ when } x\geq\frac{1}{2}\\ \leq 0 \ , \text{ when } x\leq\frac{1}{2}\end{cases} \quad ......(3.11)$$

From (3.11), we observe that, $g_{N_3N_1-N_1G}(x)$ increases in x ε $\left(0,\frac{1}{2}\right)$, and $g_{N_3N_1-N_1G}(x)$ decreases in x ε $\left(\frac{1}{2},1\right)$.

Hence the function $g_{N_3N_1-N_1G}(x)$ is CONCAVE for $x\in(0,1)$.

Now applying (3.5a) i.e.

$$mC_{f_2}(A//B;W) \leq C_{f_1}(A//B;W) \leq MC_{f_2}(A//B;W) \quad .....(3.12)$$

where $m\leq M$, together with (3.11), we get the required inequality.

for $M_{N_3N_1}(\mu_A(x_i),\mu_B(x_i);W)$ and $M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W)$,

PROPOSITION 2 : Upper bound for $M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W)$

i.e

$$M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W) \leq M_{N_3H}(\mu_A(x_i),\mu_B(x_i);W) \quad ......(3.13)$$

PROOF : Let us define :

$$g_{N_1G-N_3H}(x) = \frac{f''_{N_1G}(x)}{f''_{N_3H}(x)}, \ \forall x\in(0,1).......(3.14)$$

where $f''_{N_1G}(x)$ and $f''_{N_3H}(x)$ are taken from functional forms.

$\Rightarrow g_{N_1G-N_3H}(x) =$

$$\frac{-\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}}-\frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}}}{-\frac{1}{12\sqrt{2}}x^{-\frac{3}{2}}-\frac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}}+\left(x+\frac{1}{2}\right)^{-3}+\left(\frac{3}{2}-x\right)^{-3}} \quad ...(3.15)$$

where

Let $g_{N_1G-N_3H}(x)=\frac{P}{Q}$

$$g'_{N_1G-N_3H}(x)=\frac{QP'-PQ'}{Q^2} \quad ...(3.16)$$

Now putting the values of P',Q', and P and Q in (3.16) we observe that

$$g'_{N_1G-N_3H}(x)\begin{cases}\leq \quad 0 \quad \text{when} \quad x\leq\frac{1}{2}\\ \geq \quad 0 \quad \text{when} \quad x\geq\frac{1}{2}\end{cases} \quad ......(3.17)$$

So by first derivative test $g_{N_1G-N_3H}(x)$ is increasing in $x\in\left(0,\frac{1}{2}\right)$ and decreasing in $\left(\frac{1}{2},1\right)$ so CONCAVE in $x\in(0,1)$.

Now applying (3.12) for $M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W)$ and $M_{N_3}(\mu_A(x_i),\mu_B(x_i);W)$, we get the required inequality. i.e

$$M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W) \leq M_{N_3H}(\mu_A(x_i),\mu_B(x_i);W)$$

Now combining (3.7) and (3.13), we get the inequality (3.6) i.e.

$$M_{N_3N_1}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_1G}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_3H}(\mu_A(x_i),\mu_B(x_i);W)$$

THEOREM 3 : For $M_{N_2N_3}(\mu_A(x_i),\mu_B(x_i);W)$, $M_{N_2N_1}(\mu_A(x_i),\mu_B(x_i);W)$, $M_{N_2G}(\mu_A(x_i), \mu_B(x_i);W)$ and $M_{N_2H}(\mu_A,\mu_B;W)$, we have the following result :

$$M_{N_2N_3}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_2N_1}(\mu_A(x_i),\mu_B(x_i);W) \le \frac{1}{3}M_{N_2G}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_2H}(\mu_A(x_i),\mu_B(x_i);W) \quad \ldots\ldots\ (3.18)$$

To prove the inequality (3.18), we consider the following prepositions.

PROPOSITION 3 : The lower bound for $M_{N_2N_1}(\mu_A(x_i),\mu_B(x_i);W)$

i.e. $$M_{N_2N_3}(\mu_A(x_i),\mu_B(x_i);W) \le M_{N_2N_1}(\mu_A(x_i),\mu_B(x_i);W) \quad \ldots\ldots(3.19)$$

PROOF : Let us define

$$g_{N_2N_3-N_2N_1}(x) = \frac{f''_{N_2N_3}(x)}{f''_{N_2N_1}(x)} \quad \ldots\ldots(3.20)$$

where

$$f''_{N_2N_3}(x) = \frac{1}{8}x^{-\frac{1}{2}}\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}\left(\frac{x+\frac{1}{2}}{2}\right)^{\frac{1}{2}} - \frac{1}{32}\left\{\left(\sqrt{x}+\sqrt{\frac{1}{2}}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{3}{2}}\right\}$$
$$-\frac{1}{8}(1-x)^{-\frac{3}{2}}\left(\frac{\frac{3}{2}-x}{2}\right)^{\frac{1}{2}} - \frac{1}{32}\left(\sqrt{1-x}+\sqrt{\frac{1}{2}}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{3}{2}} + \frac{1}{12\sqrt{2}}x^{-\frac{3}{2}} + \frac{1}{12\sqrt{2}}(1-x)^{-\frac{3}{2}} \quad \ldots\ldots(3.21)$$

$$f''_{N_2N_1}(x) = N + \frac{1}{4(2x)^{\frac{3}{2}}} + \frac{1}{4[2(1-x)^{\frac{3}{2}}]} \quad \ldots\ldots(3.22)$$

and

$$N = \frac{1}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}} - \frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}} + \frac{1}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}} - \left(\frac{(1-x)}{2}\right)^2\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}} \quad \ldots(3.23)$$

Let $$g_{N_2N_3-N_2N_1}(x) = \frac{P}{Q}$$

Differentiating (3.23) w.r.t. x, we get

$$g'_{N_2N_3-N_2N_1}(x) = \frac{P'Q-PQ'}{Q^2} \quad \ldots\ldots(3.24)$$

which gives

$$g'_{N_2N_3-N_2N_1}(x) = \begin{cases} \le 0 & \text{when } x \le \frac{1}{2} \\ \ge 0 & \text{when } x \ge \frac{1}{2} \end{cases} \quad \ldots\ldots(3.25)$$

From (3.25) we conclude that $g_{N_2N_3-N_2N_1}(x)$ is increasing when $x \le \frac{1}{2}$ and decreasing when $x \ge \frac{1}{2}$. Hence $g_{N_2N_3-N_2N_1}(x)$ is CONCAVE in $x \in (0,1)$.

Now applying (3.5a) for $M_{N_2N_3}(A//B;W)$ and $M_{N_2N_1}(A//B;W)$, we get the required inequality (3.13).

PROPOSITION 4 : The upper bound for $M_{N_2N_1}(A//B;W)$ i.e. the following inequality holds good:

$$M_{N_2N_1}(A//B;W) \le \frac{1}{3}M_{N_2G}(A//B;W) \quad \ldots\ldots(3.26)$$

PROOF : We define

$$g_{N_2N_1-N_2G}(x) = \frac{f''_{N_2N_1}(x)}{f''_{N_2G}(x)}, \forall x \in (0,1)$$

$$= \frac{N + \frac{1}{4(2x)^{-\frac{3}{2}}} + \frac{1}{4[2(1-x)]^{\frac{3}{2}}}}{N + \frac{1}{16}\left(\frac{x}{2}\right)^{-\frac{3}{2}} + \frac{1}{16}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}} \quad ....(3.27)$$

where N is given by (3.23).

Now differentiating (3.27) w.r.t. x, we get

$$g'_{N_2N_1-N_2G}(x) = \frac{D\left[N' - \frac{3}{4}(2x)^{-\frac{5}{2}} + \frac{3}{4}[2(1-x)]^{-\frac{5}{2}}\right] - N\left[D' - \frac{3}{64}\left(\frac{x}{2}\right)^{-\frac{5}{2}} + \frac{3}{64}\left(\frac{1-x}{2}\right)^{-\frac{5}{2}}\right]}{D^2}$$

......(3.28)

From (3.28), we conclude that

$$g'_{N_2N_1-N_2G}(x) = \begin{cases} \leq 0 \text{ , when } x \leq \frac{1}{2} \\ \geq 0 \text{ , when } x \geq \frac{1}{2} \\ M = \sup_{x\in(0,1)} g_{N_2N_1-N_2G}(x) \\ \quad = \frac{1}{3} \end{cases} \quad ....(3.29)$$

From (3.29) and (3.28), we observe that $g_{N_2N_1-N_2G}(x)$ is increasing in $x \in \left(0, \frac{1}{2}\right)$ and decreasing in $x \in \left(\frac{1}{2}, 1\right)$. Hence CONCAVE in $x \in (0, 1)$.

Applying (3.5a) for $M_{N_2N_1}(A//B;W)$ and $M_{N_2G}(A//B;W)$ together with (3.29), we get the required inequality.

PROPOSITION 5 : The upper bound for $\frac{1}{3}M_{N_2G}(A//B;W)$ i.e. the following inequality holds good :

$$\frac{1}{3}M_{N_2G}(A//B;W) \leq M_{N_2H}(A//B;W)$$

......(3.30)

PROOF : We define

$$g_{N_2G-N_2H}(x) = \frac{f''_{N_2G}(x)}{f''_{N_2H}(x)}$$

where $f''_{N_2G}(x)$ and $f''_{N_2H}(x)$ are taken from functional forms.

$$= \frac{N + \frac{1}{16}\left(\frac{x}{2}\right)^{-\frac{3}{2}} + \frac{1}{16}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}}{D} = \frac{P}{D} \quad ......(3.31)$$

where

$$D = -\frac{1}{8\sqrt{2}}\left[\left(\sqrt{x} + \sqrt{\frac{1}{2}}\right)\left(x + \frac{1}{2}\right)^{-\frac{3}{2}} - \left(x + \frac{1}{2}\right)^{-\frac{1}{2}} x^{-\frac{1}{2}}\right]$$
$$-\frac{1}{8\sqrt{2}}\left[\left(x + \frac{1}{2}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} - x^{-\frac{1}{2}}\left(x + \frac{1}{2}\right)^{-\frac{1}{2}}\right]$$
$$-\frac{1}{8\sqrt{2}}\left[\left(\sqrt{1-x} + \sqrt{\frac{1}{2}}\right)\left(\frac{3}{2} - x\right)^{-\frac{3}{2}} + \left(\frac{3}{2} - x\right)^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\right]$$
$$-\frac{1}{8\sqrt{2}}\left[\left(\frac{3}{2} - x\right)^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}} - (1-x)^{-\frac{1}{2}}\left(\frac{3}{2} - x\right)^{-\frac{1}{2}}\right] + \frac{1}{2}\left(x + \frac{1}{2}\right)^{-2} + \frac{1}{2}\left(\frac{3}{2} - x\right)^{-2}$$

......(3.32)

Differentiating (3.31) w.r.t. x, we get

$$g'_{N_2G-N_2H}(x) = \frac{P'D - PD'}{D^2} \quad ......(3.33)$$

which gives

$$g'_{N_2G-N_2H}(x) = \begin{cases} \leq 0 \text{ , when } x \leq \frac{1}{2} \\ \geq 0 \text{ , when } x \geq \frac{1}{2} \\ M = \sup_{x\in(0,1)} g_{N_2G-N_2H}(x) \\ \quad = 3 \end{cases} \quad ....(3.34)$$

From (3.34) we conclude that $g_{N_2G-N_2H}(x)$ is increasing in $x \in \left(0, \frac{1}{2}\right)$ and decreasing in $x \in \left(\frac{1}{2}, 1\right)$. Hence CONCAVE in $x \in (0,1), \forall\, x$.

Now applying (3.5a) for $M_{N_2G}(A//B;W)$ and $M_{N_2H}(A//B;W)$ together with (3.34), we get the required inequality. Now combining (3.19), (3.26) and (3.30), we get the inequality

$$M_{N_2N_3}(A//B;W) \le M_{N_2N_1}(A//B;W) \le \frac{1}{3} M_{N_2G}(A//B;W)$$
$$\le M_{N_2H}(A//B;W)$$

PROPOSITION 6 : The inequality

$$M_{N_1G}(A//B;W) \;\le\; M_{N_1H}(A//B;W) \qquad \ldots\ldots(3.35)$$

holds good :

PROOF : Let us define

$$g_{N_1G-N_1H}(x) = \frac{f''_{N_1G}(x)}{f''_{N_1H}(x)}, \quad \forall x \in (0,1)$$

where $f''_{N_1G}(x) \;=\; -\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}}$

and

$$f''_{N_1H}(x) \;=\; -\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}} - \left(x+\frac{1}{2}\right)^{-3} - \left(\frac{3}{2}-x\right)^{-3}$$

i.e.

$$g_{N_1G-N_1H}(x) = \frac{-\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{8}(1-x)^{-\frac{3}{2}}}{-\frac{1}{8\sqrt{2}}x^{-\frac{3}{2}} - \frac{1}{8\sqrt{2}}(1-x)^{-\frac{3}{2}} - \left(x+\frac{1}{2}\right)^{-3} - \left(\frac{3}{2}-x\right)^{-3}}$$
$$= \frac{P}{Q} \qquad (3.36)$$

Differentiating (3.36) w.r.t. *x*, we get

$$g'_{N_1G-N_1H}(x) \;=\; \frac{P'Q - PQ'}{Q^2} \qquad \ldots\ldots(3.37)$$

which gives

$$g'_{N_1G-N_1H}(x) = \begin{cases} \le\; 0\;, & \text{when } \; x \le \frac{1}{2} \\ \ge\; 0\;, & \text{when } \; x \ge \frac{1}{2} \end{cases} \qquad \ldots\ldots(3.38)$$

From (3.38), we conclude that $g_{N_1G-N_1H}(x)$ is increasing in $x \in \left(0, \frac{1}{2}\right)$ and decreasing in $x \in \left(\frac{1}{2}, 1\right)$. Hence CONCAVE in $x \in (0,1)$.

Applying (3.5a) together with (3.38), we get the required inequality (3.35).

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