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Simultaneous Equivariant Estimation of the Parameters of a Location-Scale Model Based on General Progressive Type II Right Censored Sample

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ABSTRACT

In this paper, by assuming that a general progressive Type II right censored sample is available, we obtain Quadratic-type Minimum Risk Equivariant (QA-MRE) estimators for the vector parameters of (ξ, τ)′ and (ξ, τ²)′ based on the general Type II progressive right censored sample. Further Minimum Risk Equivariant (MRE) estimator of (ξ, τ)′ is obtained with respect to Linex type loss function. These generalize the results of Chandrasekar et.al. (2002) for progressive Type II right censored sample . The paper is organized as follows : Section 2 deals with the problem of QA-MRE estimators for the vector parameters. In the last Section, we consider the problem of simultaneous equivariant estimation of the parameters under Linex loss function (Varian,1975).

1. Introduction

Progressive Type II right censored sampling is an important method of obtaining data in life-testing studies. As pointed out by Aggarwala and Balakrishnan (1998), the scheme of progressive censoring enables us to use live units, removed early, in other tests. Balakrishnan and Sandhu (1996), by assuming a general progressive Type II right censored sample, derived the BLUE's for the parameters of one-and two-parameter exponential distributions. For the later, they also derived MLE's and shown that they are simply the BLUE's, adjusted for their bias.

Let us consider the following general progressive Type II right censoring scheme (Balakrishnan and Sandhu, 1996) : Suppose N randomly selected units were placed on a life test; the failure times of the first r units to fail were not observed ; at the time of the (r+1)-th failure, R_{r+1} number of surviving units are withdrawn from the test randomly, and so on; at the time of the (r+i)-th failure, R_{r+i} number of surviving units are randomly withdrawn from the test ; finally, at the time of the n-th failure, the remaining $R_n = N - n - R_{r+1} - R_{r+2} - \dots - R_{n-1}$ are withdrawn from the test. Suppose $X_{r+1:N} \le X_{r+2:N} \le \dots \le X_{n:N}$ are the life-times of the completely observed units to fail, and $R_{r+1}, R_{r+2}, \dots, R_n$ are the number of units withdrawn from the test at these failure times, respectively. It follows that $N = n + \sum_{i=r+1}^{n} Ri$

If the failure times are from a continuous population with the pdf f and the distribution function F, then the joint pdf of ($X_{r+1:N}, X_{r+2:N}, \dots, X_{n:N}$) is given by

$$g_\theta(x_{r+1}, \dots, x_n) = c\,\{F_\theta(x_{r+1})\}^r \prod_{i=r+1}^{n} f_\theta(x_i)\{1-F_\theta(x_i)\}^{R_i} \quad , \tag{1.1}$$

where

$$c = \binom{N}{r}(N-r)\prod_{j=r+2}^{n}\left(N - \sum_{i=r+1}^{j-1} R_i - j + 1\right) \quad .$$

In this paper, by assuming that such a general progressive type II censored sample is available from exponential distribution, we obtain Q_A-MRE estimators for the vector parameters (ξ, τ)′ and (ξ, τ²)′. Further MRE estimator (ξ, τ)′ is obtained with respect to Linex loss function.

Exponential location- scale model

In this case, the common pdf is taken to be

$$f_\theta(x) = \begin{cases} \frac{1}{\tau} e^{-\frac{(x-\xi)}{\tau}} & ; x > \xi \, ; x \varepsilon \mathbf{R}, \tau > 0, \\ 0 & , \text{otherwise.} \end{cases}$$

Note that $\theta = \{ \xi , \tau)'$. Then (1.1) reduces to

$$g_\theta(x_{r+1}, \dots, x_n) = c\frac{1}{\tau^{n-r}}\left\{1 - e^{\left[-(x_{r+1}-\xi)/\tau\right]}\right\}^r e^{\left\{-\sum_{i=r+1}^{n}(R_i+1)(x_i-\xi)/\tau\right\}} , \tag{1.2}$$

$\xi < x_{r+1:N} < \dots < x_{n:N} < \infty \, ; \, \xi \in \mathbf{R}, \tau > 0$.

Thus the joint distribution of ($X_{r+1:N}, \dots, X_{n:N}$) belongs to a location-scale family with the location-scale parameter θ = (ξ, τ)′.

2 . Q_A–MRE

Following Edwin Prabakaran and Chandrasekar (1994), we first obtain the Q_A-MRE estimator of (ξ, τ)′. Take $\delta_{01}(\mathbf{X}) = X_{r+1:N}$ and

$$\delta_{02}(\mathbf{X}) = \sum_{i=r+2}^{n} (R_i+1)\,(X_{i:N} - X_{r+1:N}) .$$

Note that $\boldsymbol{\delta}_0(\mathbf{X}) = (\,\delta_{01}(\mathbf{X}), \delta_{02}(\mathbf{X})\,)'$ is an equivariant estimator but not complete sufficient. Since we are interested in the evaluation of conditional distribution under (ξ, τ)′ = (0, 1)′, we take (ξ, τ)′ = (0,1)′ in (1.2).In order to find (w_1^*, w_2^*) as given in (2.1) ,

$$w^* = \frac{E(\delta_{01}\delta_{02} \,|\, z)}{E(\delta_{02}^2 \,|\, z)} \quad \text{and} \quad \frac{1}{w^*} = \frac{E(\delta_{02} \,|\, z)}{E(\delta_{02}^2 \,|\, z)} , \tag{2.1}$$

consider the transformation $Z_{r+1} = X_{r+1:N}$

$$Z_{r+2} = \sum_{i=r+2}^{n} (R_i + 1)\,(X_{i:N} - X_{r+1:N})$$

$$\text{and } Z_j = \frac{X_{j:N} - X_{r+1:N}}{\sum_{i=r+2}^{n}(R_i+1)(X_{i:N} - X_{r+1:N})} , \quad j = r+3, \dots, n$$

Thus the joint pdf of ($Z_{r+1}, \dots, Z_n$) is given by

$$h(z_{r+1}, \dots, z_n) = \frac{c}{(R_{r+2}+1)} z_{r+2}^{n-r-2}\left\{1 - e^{(-z_{r+1})}\right\}^r e^{-\sum_{i=r+1}^{n}(R_i+1)z_{r+1}+z_{r+2}} \qquad 0 \le z_{r+1} \le z_{r+2} \le \dots \le z_n < \infty .$$

Also, the joint pdf of ($Z_{r+3}, \dots, Z_n$) is given by

$$h_1(z_{r+3}, \dots, z_n) = \frac{c}{(R_{r+2}+1)} \int_0^\infty\!\!\int_0^\infty z_{r+2}^{n-r-2}\left\{1 - e^{-z_{r+1}}\right\}^r e^{-\sum_{i=r+1}^{n}(R_i+1)z_{r+1}+z_{r+2}} \, dz_{r+1} dz_{r+2}$$

$$= \frac{c}{(R_{r+2}+1)}\Gamma(n-r-1)B_1\left(\sum_{i=r+1}^{n}(R_i+1), r+1\right) .$$

Thus the conditional pdf of $(\delta_{01}, \delta_{02}) = (Z_{r+1}, Z_{r+2})$ given $(Z_{r+3}, \dots, Z_n)$ is given by

$$h_2((z_{r+1}, z_{r+2}) \,|\, z_{r+3}, \dots, z_n) = \frac{z_{r+2}^{n-r-2}\left\{1 - e^{-z_{r+1}}\right\}^r e^{-\sum_{i=r+1}^{n}(R_i+1)z_{r+1}+z_{r+2}}}{B_1\left(\sum_{i=r+1}^{n}(R_i+1), r+1\right)\Gamma(n-r-1)} \qquad 0 \le z_{r+1} < \infty, \ \ 0 \le z_{r+2} < \infty .$$

It may be noted that $(\delta_{01}, \delta_{02})'$ is independent of $(Z_{r+3}, \dots, Z_n)'$.

Thus $E(Z_{r+1} Z_{r+2}) = (n-r-1)\,E(-\log U)$,

where $U \sim B_1\left(\sum_{i=r+1}^{n}(R_i+1), r+1\right)$.

Also, the conditional pdf of Z_{r+2} given $(Z_{r+3}, \dots, Z_n)'$ is same as the marginal pdf of Z_{r+2} and is given by

$$h_3(z_{r+2}) = \frac{z_{r+2}^{n-r-2}\, e^{-z_{r+2}}}{\Gamma(n-r-1)} .$$

Now

$$E(z_{r+2} \,|\, z_{r+3}, \dots, z_n) = \frac{1}{\Gamma(n-r-1)}\int_0^\infty z_{r+2}^{n-r-2} e^{-z_{r+2}} dz_{r+2} = (n-r-1)$$

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(2.2) and $E\left(Z_{r+2}^{2}\,|\,Z_{r+3},\ldots,Z_n\right)=(n-r)(n-r-1)$.
(2.3)

Then

$w_1^* = \dfrac{E(-\log U)}{(n-r)}$ and $w_2^* = \dfrac{1}{(n-r)}$, in view of (2.2) and (2.3),

where $U \sim B_1\left(\sum_{i=r+1}^{n}(R_i+1),\, r+1\right)$.

Therefore the MRE estimator of $(\xi, \tau)'$ is given by

$$\delta_1^*(X) = X_{r+1:N} - \frac{E(-\log U)}{(n-r)}\sum_{i=r+2}^{n}(R_i+1)(X_{i:N}-X_{r+1:N}) \quad\text{and}\quad \delta_2^*(X) = \frac{\sum_{i=r+2}^{n}(R_i+1)(X_{i:N}-X_{r+1:N})}{(n-r)},$$

where $U \sim B_1\left(\sum_{i=r+1}^{n}(R_i+1),\, r+1\right)$ and $E(-\log U) = \sum_{j=0}^{r}\dfrac{1}{\sum_{i=r+1}^{n}(R_i+1)+j}$.

Remark 2.1. If $r = 0$ and $R_i = r_i$, then the above estimators reduce to

$$\delta_1^*(\mathbf{X}) = X_{1:N} - (1/Nn)\sum_{i=2}^{n}(r_i+1)\,(X_{i:N}-X_{r+1:N}) \quad\text{and}\quad \delta_2^*(X) = \frac{\sum_{i=2}^{n}(r_i+1)(X_{i:N}-X_{1:N})}{n} ,$$

which are same as the Type II progressive right censored sample (Chandrasekar et.al. 2002)

Now let us consider the problem of estimating $(\xi, \tau^2)'$. Define $\delta_0(\mathbf{X}) = (\delta_{01}(\mathbf{X}), \delta_{02}(\mathbf{X}))'$, where $\delta_{01}(\mathbf{X}) = X_{r+1:N}$ and

$$\delta_{02}(\mathbf{X}) = \left\{\sum_{i=r+2}^{n}(R_i+1)(X_{i:N}-X_{r+1:N})\right\}^2 .$$

we have

$$w_1^* = \frac{\begin{array}{c}[a_{11}a_{22}E(\delta_{02}^2\,|\,\mathbf{z})E(\delta_{01}g\,|\,\mathbf{z}) - a_{12}^2E(\delta_{02}g\,|\,\mathbf{z})E(\delta_{01}\delta_{02}\,|\,\mathbf{z})\\ -a_{12}a_{22}\{E(\delta_{02}^2\,|\,\mathbf{z})E(g\,|\,\mathbf{z}) - E(\delta_{02}g\,|\,\mathbf{z})E(\delta_{02}\,|\,\mathbf{z})\}]\end{array}}{a_{11}a_{22}E(g^2\,|\,\mathbf{z})E(\delta_{02}^2\,|\,\mathbf{z}) - a_{12}^2E^2(\delta_{02}g\,|\,\mathbf{z})} \qquad (2.$$

and

$$\frac{1}{w_2^*} = \frac{\begin{array}{c}[-a_{11}a_{22}\{E(g^2\,|\,\mathbf{z})E(\delta_{01}\delta_{02}\,|\,\mathbf{z}) - E(\delta_{02}g\,|\,\mathbf{z})E(\delta_{01}g\,|\,\mathbf{z})\}\\ +a_{11}a_{22}E(g^2\,|\,\mathbf{z})E(\delta_{02}\,|\,\mathbf{z}) - a_{12}^2E(\delta_{02}g\,|\,\mathbf{z})E(g\,|\,\mathbf{z})]\end{array}}{a_{11}a_{22}E(g^2\,|\,\mathbf{z})E(\delta_{02}^2\,|\,\mathbf{z}) - a_{12}^2E^2(\delta_{02}g\,|\,\mathbf{z})} \qquad (2.$$

Thus the MRE estimator is (ξ,τ^m) is given by

$$\delta_1^* = \partial_{01} - gw_1^* \quad\text{and}\quad \delta_2^* = \partial_{02}/w_2^* \qquad (2.6)$$

Taking $g(\mathbf{X}) = \sum_{i=r+2}^{n}(R_i+1)(X_{i:N}-X_{r+1:N})$, from equations (2.4), (2.5) and (2.6), we obtain

$$w_1^* = \frac{E(-\log U)\{a_{11}a_{22}(n-r+2) - a_{12}^2(n-r)\} - 2a_{12}a_{22}}{(n-r)\{a_{11}a_{22}(n-r+2) - a_{12}^2(n-r+1)\}} \quad\text{and}$$

$$\frac{1}{w_2^*} = \frac{E(-\log U)a_{11}a_{22} + \{a_{11}a_{22}(n-r) - a_{12}^2(n-r+1)\}}{(n-r)(n-r+1)\{a_{11}a_{22}(n-r+2) - a_{12}^2(n-r+1)\}}$$

Therefore the MRE estimator of $(\xi, \tau^2)'$ given by

$$\delta_1^* = X_{r+1:N} - \sum_{i=r+2}^{n}(R_i+1)\,(X_{i:N}-X_{r+1:N})\,w_1^* \quad\text{and}\quad \delta_2^* = \frac{\sum_{i=r+2}^{n}(R_i+1)(X_{i:N}-X_{r+1:N})}{w_2^*} .$$

Remark 2.2 If $r = 0$ and $R_i = r_i$, then the above estimators reduce to that of exponential location-scale model (Chandrasekar et.al. 2002).

3. Linex loss function

Following Varian (1975), let us find the MRE estimator of $(\xi, \tau)'$, under Linex loss function. Consider the location-scale invariant loss function

$$L(\xi,\tau;\delta) = e^{a(\delta_1-\xi)/\tau} - a(\delta_1-\xi)/\tau - 1 + e^{b(\delta_2/\tau-1)} - b(\delta_2/\tau-1) - 1, \quad a \in \mathbf{R}-\{0\},\ b>0 .$$

In order to find (w_1^*, w_2^*), consider

$$R(\boldsymbol{\delta}\,|\,\mathbf{z}) = E\left[\{e^{a(\delta_{01}-gw_1)} - a(\delta_{01}-gw_1) - 1 + e^{b(\delta_{02}/w_2-1)} - b(\delta_{02}/w_2-1) - 1\}\,|\,\mathbf{z}\right],$$

where $\delta_{01} = X_{r+1:N}$ and $\delta_{02} = \sum_{i=r+2}^{n}(R_i+1)(X_{i:N}-X_{r+1:N})$ That is

$$\begin{aligned}R(\boldsymbol{\delta}\,|\,\mathbf{z}) &= E\left[(e^{a\delta_{01}}\,e^{-aw_1\delta_{02}})\,|\,\mathbf{z}\right] - a\,E(\delta_{01}\,|\,\mathbf{z})\\ &\quad + aw_1E(\delta_{02}\,|\,\mathbf{z}) - 1 + e^{-b}E(e^{b/w_2\,\delta_{02}}\,|\,\mathbf{z})\\ &\quad + b - b/w_2\,E(\delta_{02}\,|\,\mathbf{z}) - 1\end{aligned}$$

$$\begin{aligned}&= E(e^{a\delta_{01}})\,E(e^{-aw_1\delta_{02}}) - a\,E(\delta_{01})\\ &\quad + a\,w_1E(\delta_{02}) - 1 + e^{-b}E(e^{b/w_2\,\delta_{02}})\\ &\quad + b - b/w_2\,E(\delta_{02}) - 1,\end{aligned}$$

since $(\delta_{01}, \delta_{02})'$ is independent of $\mathbf{Z} = (Z_{r+3},\ldots,Z_n)'$ δ_{01} and δ_{02} are independent and also δ_{01} is independent of $\mathbf{Z}$ and δ_{02} is independent of $\mathbf{Z}$. Thus

$$\begin{aligned}R(\delta\,|\,z) &= \frac{B_1\left(\sum_{i=r+1}^{n}(R_i+1)-a,\ r+1\right)}{B_1\left(\sum_{i=r+1}^{n}(R_i+1),\ r+1\right)}\frac{1}{(1+aw_1)^{n-r-1}}\\ &\quad - aE(-\log U) + aw_1(n-r-1) - 1\\ &\quad + e^{-b}\frac{1}{(1-b/w_2)^{n-r-1}} + b - b/w_2(n-r-1) - 1 ,\end{aligned}$$

where $U \sim B_1\left(\sum_{i=r+1}^{n}(R_i+1),\, r+1\right)$.

Define $$k = \frac{B_1\left(\sum_{i=r+1}^{n}(R_i+1)-a,\ r+1\right)}{B_1\left(\sum_{i=r+1}^{n}(R_i+1),\ r+1\right)}$$

Moreover,

$$\frac{\partial R}{\partial w_1} = k(-(n-r-1)(1+aw_1)^{-(n-r-1)-1}\,a + a(n-r-1)$$

(3.1)

$$\frac{\partial R}{\partial w_2} = e^{-b}(-(n-r-1))(1-b/w_2)^{-(n-r-1)-1}\,(b/w_2^2) + b(n-r-1)/w_2^2$$

(3.2)

and $\frac{\partial^2 R}{\partial w_1 \partial w_2} = 0 = \frac{\partial^2 R}{\partial w_2 \partial w_1}$

From (3.1), we obtain $w_1^* = \frac{(1/k)^{-1/(n-r)} - 1}{a}$, $a \in \mathbf{R} - \{0\}$

and from (3.2), we obtain $w_2^* = \frac{b}{\left(1 - e^{-b/(n-r)}\right)}$, $b > 0$

.

Therefore the MRE estimator of $(\xi, \tau)'$ is given by

$$\delta_1^* = X_{r+1:N} - \sum_{i=r+2}^{n} (R_i + 1)\left(X_{i:N} - X_{r+1:N}\right) w_1^* \quad \text{and} \quad \delta_2^* = \frac{\sum_{i=r+2}^{n} (R_i + 1)(X_{i:N} - X_{r+1:N})}{w_2^*}$$

Remark 3.1. If $r = 0$ and $R_i = r_i$, then the above estimators reduce to that of Linex loss function discussed in Leo Alexander (2000).

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