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Research Paper

Some Fixed Point Theorems For Cyclic Contractions In Dislocated Quasi-Metric Spaces

Mathematics

KEYWORDS : cyclic map, cyclical contraction, Common fixed point, dislocated quasi-metric

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ABSTRACT

In this paper, we study some fixed point theorems for cyclic contractions in a dislocated quasi-metric space which improve and extend some existing results.

1. Introduction

Notion of dislocated metric spaces was introduced by Hitzler and Seda in 2000 as a generalization of metric space. They generalized the Banach Contraction Principle in such spaces. These metrics play a very important role not only in topology but also in other branches of science involving mathematics especially in logic programming and electronic engineering. Fixed point theory has been a subject of growing interest of many researchers for various types of well known contractions in these spaces. In 2003 Kirk et al [8] introduced cyclic contractions in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings, Since then many authors has given results in this field. In this paper we extend and unify existing results in the recent literature.

Definition 1.1 [7, 10]: Let X be a non-empty and let $d: X \times X \to \mathbb{R}^+$ be a function, called a distance function if for all, $x, y, z \in X$, and satisfies:

1. $For\ all\ x, y \in X, if\ d(x,x) = 0,$
2. $For\ all\ x, y \in X, if\ d(x,y) = d(y,x) = 0, then\ x = y,$
3. $For\ all\ x, y \in X, if\ d(x,y) = d(y,x),$
4. $For\ all\ x, y, z \in X, d(x,y) \leq d(x,z) + d(z,y).$

If d satisfies conditions (1) to (4), then it is called a metric. If it satisfies conditions (1), (2) and (4), it is called a quasi-metric. If it satisfies (2), (3) and (4), we will call it a dislocated metric (Or simply d-metric). If it satisfies conditions (2) and (4), it is called a dislocated quasi-metric (or simply dq-metric).

Definition 1.2[10]: A sequence (x_n)in dislocated quasi-converges (for short dq-converges) to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.In this case x is called dq-limit of(x_n).

Definition 1.3[10]: A sequence (x_n) in dq-metric space (X, d) is called Cauchy if for all $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0, d(x_m, x_n) < \epsilon, d(x_n, x_m) < \epsilon$.

Definition 1.4[20]: A dq-metric space (X, d) is called complete if every Cauchy sequences in it is dq-convergent.

Lemma 1.5[10]: Every subsequence of dq-convergent sequence to a point $x_0$is dq-convergent to x_0.

Definition 1.6[10]: let (X, d) be a dq-metric spaces. A map $f: X \to X$ is called contraction if there exists $0 \leq \lambda \leq 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$, for all $x, y \in X$.

Lemma 1.7[10]: dq-limit in a -metric space is unique.

Definition 1.8[3]: Let A and B be nonempty subsets of a metric space (X, d) and

$T: A \cup B \to A \cup B$. T is called a cyclic map iff and $T(A) \subseteq B \; and \; T(B) \subseteq A$.

Definition 1.10 [8]: Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T: A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y), \; forall \; x \in A \; and \; y \in B$.

Definition 1.11[2]: Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T: A \cup B \to A \cup B$ is called a Kannan type cyclic contraction, if there exist $k \in \left(0, \frac{1}{2}\right)$ such that $d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)], forall \; x \in A \; and \; y \in B$.

In [2] Karapinar et al has been shown that Kannan type cyclic contraction and cyclic contraction are independent of each other.

Definition 1.12[2]: Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T: A \cup B \to A \cup B$ is called a Chatterjee type cyclic contraction, if there exist $k \in \left(0, \frac{1}{2}\right)$ such that $d(Tx, Ty) \leq k \; max[d(x, x), d(Tx, x), d(Ty, y)], \; forall \; x \in A \; and \; y \in B$.

Definition 1.13[8]: Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T: A \cup B \to A \cup B$ is called a d-type cyclic contraction if there exist $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) \; forall \; x \in A \; and \; y \in B$.

2. Main Results

Theorem 2.1: Let A and B be nonempty subsets of a complete dislocated quasi-metric space (X, d). Let $T: A \cup B \to A \cup B$ be a cyclic mapping that satisfies the condition

$$d(Tx,Ty) \leq \propto max \begin{bmatrix} d(x,y), d(Tx,x), d(Ty,y), d(Tx,Ty), \\ d(x,x), d(Ty,Ty) \end{bmatrix} \quad (1)$$

$forall\ x \in A\ and\ y \in B, where\ \propto \in [0,1)$. Then T has a unique fixed point in $A \cap B$.

Proof: Taking a point $x \in A$ and using contractive condition of theorem, we have

$$d(T^2x, Tx) = d(T(Tx), Tx)$$

$$d(T^2x, Tx) = d(T(Tx), Tx) \leq \propto \max \begin{Bmatrix} d(Tx,x), d(T(Tx),Tx), d(Tx,x), d(T(Tx),Tx), \\ d(Tx,Tx), d(Tx,Tx) \end{Bmatrix}$$

$$\leq \propto \max \begin{Bmatrix} d(Tx,x), d(T^2x,Tx), d(Tx,x), d(T^2x,Tx), \\ d(Tx,Tx), d(Tx,Tx) \end{Bmatrix}$$

$$\leq \propto \max \begin{Bmatrix} d(Tx,x), d(T^2x,Tx), d(Tx,x), d(T^2x,Tx), \\ d(Tx,Tx), d(Tx,Tx) \end{Bmatrix}$$

$$\leq \propto \max\{d(Tx,x), d(Tx,Tx)\} \leq \propto \max\{d(Tx,x), 2d(Tx,x)\}$$

$$\leq \beta d(Tx,x), where\ \beta = 2 \propto < 1$$

Now by induction we have, $d(T^{n+1}x, T^n x) \leq \beta^n d(Tx,x)$.

Let $n, m \in N\ with\ m > n$, using the triangular inequality, we obtain that:

$$d(T^m x, T^n x) \leq d(T^m x, T^{m-1}x) + d(T^{m-1}x, T^{m-2}x) + \cdots + d(T^{n+1}x, T^n x)$$

$$\leq [\ \beta^n + \beta^{n+1} + \cdots + \beta^{m-1}] d(Tx,x) \leq \frac{\beta^n}{1-\beta}\ d(Tx,x)$$

Since $\beta < 1, \beta^n \to 0\ as\ n \to \infty, we\ get\ d(T^m x, T^n x) \to 0.$

$Thus\ T^n x\ is\ a\ Cauchy\ sequence.$

Since (X, d) is complete, we have $T^n x$ dq-converges to some $z \in X$. We note that $T^{2n}x$ is a sequence in A and $T^{2n-1}x$ is a sequence in B in a way that both sequences tend to same limit it z. Since A and B are closed have that $z \in A \cap B$. Hence $A \cap B \neq \emptyset$.

We claim that $Tz = z$. Considering the condition (1) we have:

$$d(Tz,z) = d(Tz,T^{2n}x) \leq\propto max\begin{bmatrix} d(z,T^{2n-1}x), d(Tz,z), d(T^{2n}x,T^{2n-1}x), d(Tz,T^{2n}x), \\ d(z,z), d(Tz,Tz), d(T^{2n}x,T^{2n}x) \end{bmatrix}$$

Taking limit as $n \to \infty$ in above inequality, we have

$d(Tz,z) \leq\propto \max[\, d(Tz,z), 2(Tz,z), 2(Tz,z)]$

$d(Tz,z) \leq 2 \propto d(Tz,z) \leq \mathbb{Q}d(Tz,z), where\ \mathbb{Q} = 2 \propto$

Which is a contradiction, hence $d(Tz,z) = 0 \Rightarrow Tz = z$

Uniqueness:

Let us assume that two fixed point's u and v i.e. $Tu = u\ and\ Tv = v$.

$$d(Tu,Tv) \leq\propto max\begin{bmatrix} d(u,v), d(Tu,u), d(Tv,v), d(Tu,Tv), \\ d(u,u), d(Tu,Tu), d(Tv,Tv) \end{bmatrix}$$

$d(Tu,Tv) \leq\propto d(u,v)$, This implies that $u = v$.

This completes the proof of the theorem.

Theorem 2.2: Let A and B be nonempty subsets of a complete dislocated quasi-metric space (X, d). Let $T: A \cup B \to A \cup B$ be a cyclic mapping that satisfies the condition

$$d(Tx,Ty) \leq \varphi\begin{bmatrix} d(x,y) + d(Tx,x) + d(Ty,y) + d(Tx,Ty) \\ +d(x,x) + d(Tx,Tx) + d(Ty,Ty) \end{bmatrix} \quad (2)$$

$forall\ x \in A\ and\ y \in B, where\ \varphi \in [0,1)$. Then T has a unique fixed point in $A \cap B$.

Proof: Taking a point $x \in A$ and using contractive condition of theorem, we have

$d(T^2x,Tx) = d(T(Tx),Tx)$

$$d(T^2x,Tx) \leq \varphi\begin{bmatrix} d(Tx,x) + d(T(Tx),Tx) + d(Tx,x) + d(T(Tx),Tx) \\ +d(Tx,Tx) + d(T(Tx),T(Tx)) + d(Tx,Tx) \end{bmatrix}$$

$$d(T^2x,Tx) \leq \varphi\begin{bmatrix} d(Tx,x) + d(T^2x,Tx) + d(Tx,x) + d(T^2x,Tx) \\ +d(Tx,Tx) + d(T^2x,T^2x) + d(Tx,Tx) \end{bmatrix}$$

$$d(T^2x,Tx) \leq \varphi\begin{bmatrix} d(Tx,x) + d(T^2x,Tx) + d(Tx,x) + d(T^2x,Tx) \\ +2d(Tx,x) + 2d(T^2x,Tx) + 2d(Tx,x) \end{bmatrix}$$

$d(T^2x,Tx) \leq \varphi[6d(Tx,x) + 4d(T^2x,Tx)]$

$(1 - 4\varphi)d(T^2x,Tx) \leq 6\varphi d(Tx,x)$

$d(T^2x,Tx) \le \frac{6\varphi}{(1-4\varphi)} d(Tx,x) \le \Theta d(Tx,x), \quad where\ \Theta = \frac{6\varphi}{(1-4\varphi)} < 1$

Now by induction we have, $d(T^{n+1}x, T^n x) \le \Theta^n d(Tx,x)$.

Let $n, m \in N\ with\ m > n$, using the triangular inequality,

We obtain that:

$$d(T^m x, T^n x) \le d(T^m x, T^{m-1}x) + d(T^{m-1}x, T^{m-2}x) + \cdots + d(T^{n+1}x, T^n x)$$

$$\le [\Theta^m + \Theta^{m-1} + \cdots + \Theta^n] d(Tx,x) \le \frac{\Theta^n}{1-\Theta} d(Tx,x)$$

Since $\Theta < 1, \Theta^n \to 0\ as\ n \to \infty, we\ get\ d(T^m x, T^n x) \to 0$.

$Thus\ T^n x\ is\ a\ Cauchy\ sequence.$

Since (X, d) is complete, we have $T^n x$ dq-converges to some $z \in X$. We note that $T^{2n}x$ is a sequence in A and $T^{2n-1}x$ is a sequence in B in a way that both sequences tend to same limit it z. Since A and B are closed have that $z \in A \cap B$. Hence $A \cap B \neq \emptyset$. We claim that $Tz = z$.

Considering the condition (1) we have:

$$d(Tz,z) = d(Tz, T^{2n}x) \le \varphi \begin{bmatrix} d(z, T^{2n-1}x) + d(Tz,z) + d(T^{2n}x, T^{2n-1}x) + d(Tz, T^{2n}x) \\ +d(z,z) + d(Tz,Tz) + d(T^{2n}x, T^{2n}x) \end{bmatrix}$$

Taking limit as $n \to \infty$ in above inequality, we have

$d(Tz,z) \le \varphi[\ d(Tz,z) + 2d(Tz,z) + 2d(Tz,z)]$

$d(Tz,z) \le 5\varphi d(Tz,z)$, Which is a contradiction, hence $d(Tz,z) = 0 \Rightarrow Tz = z$

Uniqueness: Let us assume that two fixed point's u and v i.e. $Tu = u\ and\ Tv = v$.

$$d(Tx,Ty) \le \varphi \begin{bmatrix} d(x,y) + d(Tx,x) + d(Ty,y) + d(Tx,Ty) \\ +d(x,x) + d(Tx,Tx) + d(Ty,Ty) \end{bmatrix}$$

$$d(Tu,Tv) \le \varphi \begin{bmatrix} d(u,v) + d(Tu,u) + d(Tv,v) + d(Tu,Tv) \\ +d(u,u) + d(Tu,Tu) + d(Tv,Tv) \end{bmatrix}$$

$$d(Tu,Tv) \le \varphi \begin{bmatrix} d(u,v) + d(u,u) + d(v,v) + d(u,v) \\ +d(u,u) + d(u,u) + d(v,v) \end{bmatrix}$$

$d(Tu,Tv) \le 2\varphi d(u,v) \le \propto d(u,v), \quad Where\ 2\varphi = \propto$

This implies that $u = v$.

This completes the proof of the theorem.

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