Mathematics

Sub-Riemannian Structures on Manifolds

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ABSTRACT *Sub-Riemannian structures naturally occur in different branches of mathematics in the study of constrained systems in classical mechanics, in optimal control, geometric measure theory and differential geometry. Let M be a smooth n-dimensional manifold and let F(M) denote the coframe bundle of M. It is a principal KL(n, R)-bundle over M, where KL(n, R) action is given by the change of basis matrices. Let G be a subgroup of KL(n, R). The K-structure on M is a principal K- subbundle of the coframe bundle of M. In the paper we introduce some notions and results from the theory of K-structures on manifolds.*

1. INTRODUCTION:

Sub-Riemannian structures naturally occur in different branches of mathematics in the study of constrained systems in classical mechanics, in optimal control, geometric measure theory and differential geometry. We define a topological manifold as follows:

Definition 1.1: A topological space M is a manifold of dimension n if

    i.   M is Housdorff, and

    ii.   M is second countable, and

    iii.   M is locally Euclidean of dimension n.

Definition 1.2: A coordinate chart on M is a pair (U, ϕ) where $U \subseteq M$ is open and $\phi : U \rightarrow \phi(U) \subseteq R^{\alpha}$ is a homeomorphism. The set U is called a coordinate domain or coordinate neighborhood or coordinate patch. If $\phi(U)$ is a ball in R^{α}; U is called a coordinate ball. A coordinate chart (U, ϕ) is centered at p if $\phi(p) = 0$.

From the Darboux theorem we know that every two contact manifolds of the dimension are locally equivalent. Sub-Riemannian structures already have local invariants has been shown in [2], when n = 1there are functions K and the eigen value of the torsion matrix which do not change under the local automorphisms of a sub-Riemannian structure. Essentially the same invariants for sub-Riemannian structures on 3-manifolds were defined in [1]. In the papers [8] and [6], similar invariants were considered and further generalized to the case of contact metric manifolds in [7]. Note that all these

invariants coincide when the dimension of a manifold is three.

A sympletic manifold (M, ω) is a smooth manifold M together with a smooth, closed, nondegenerate 2-form ω called the sympletic form. Note that the nondegeneracy condition on ω requires that M is even-dimensional. A Hamiltonian on M is a smooth function $H : M \rightarrow R$. To every Hamiltonian H on M there is associated a vector field ξ_H on M, defined by the condition $dH(v) = \omega(v, \xi_h)$ for all $v \in TM$.

2. SUB-RIEMANNIAN *K*-STRUCTURES ON MANIFOLDS:

In the present section we will introduce some notions and results from the theory of *K*-Structures. For undefined terms in this paper reader may refer to [5].

Definition 2.1: Let M be a smooth n-dimensional manifold and let F(M) denote the coframe bundle of M. It is a principal $KL(n, R)$-bundle over M ,where $KL(n, R)$ action is given by the change of basis matrices. Let G be a subgroup of $KL(n, R)$.

Definition 2.2: The *K*-structure on M is a principal *K*- subbundle of the coframe bundle of M.

Let φ be a diffeomorphism between manifolds M_1 and M_2. The map φ^* induces an isomorphism of the coframe bundles $F(M_1)$ and $F(M_2)$.

Definition 2.3: Let B^1 and B^2 be two *K*-structures on M_1 and M_2 respectively. A diffeomorphism φ is called an isomorphism of *K*-structures if $\varphi^* (B^2 = B^1)$. We will say two *K*-structures are isomorphic if there is an isomorphism between them.

Definition 2.4: Two *K*-structures B^1 and B^2 are called locally equivalent at (x,y) where $x \in M_1$ and $y \in M_2$ if there are neighborhoods U(x) and V(y) and a diffeomorphism $\varphi : U(x) \rightarrow V(y)$ such that $\varphi^*(B^2|_{V(y)}) = B^1|_{U(x`)}$.

3. RESULTS:

Assume that M is a $(2n + 1)$-dimensional manifold and D is a contact structure on M.

Definition 3.1: A triple $S=(M, D, g)$ where g is a fibrewise inner product on D is called a sub-Riemannian structure on M.

Each sub-Riemannian structure defines a natural *k*-structure consisting of adapted coframes.

Definition 3.2: We say that a coframe $\eta_1, \eta_2, \dots, \eta_{2n+1}$ of convectors in T_pM is adopted to S if $Ker(\eta_{2n+1}) = D$ and $g = \sum_{i=1}^{2n} \eta_i \otimes \eta_i$.

We may now consider the set B of all coframes adapted to S. Every sub-Riemannian structures S defines a *k*-structure B of adopted coframes, and conversely every *k*-structure B with G above gives rise to sub-Riemannian structure S.

Let (M, D) be a $(2n+1)$- dimensional contact manifold and let η be some fixed contact 1-form of D. There is a unique vector field ξ such that $\eta(\xi) = 1$, $i_\xi d_\eta = 0$.

If g is a Riemannian metric on M and φ is a (1, 1)-tensor field such that

$g(\xi, X) = \eta(X)$, $g(X, \varphi Y) = d\eta(X, Y)$, $\varphi_\varphi X = -X + \eta(X)\xi$ the metric is called the metric associated with η and

the tuple (M, η, ξ, g, φ)- a contact Riemannian manifold.

Proposition 3.3: If dim(M) = 3 then for every sub-Riemannian structure S on M there is a canonical Riemannian metric associated with g.

Proof : Let (D, g) be a sub-Riemannian structure on M. Fix a basis (e_1, e_2, v) in V. The space $E = \frac{\text{Hom}(V \wedge V, V)}{A\left(\text{Hom}(V, g)\right)}$ is 1-dimensional and is generated by the vector $e_1^* \wedge e_2^* \otimes$ v and the group G acts on E by $\begin{pmatrix} A & b \\ 0 & c \end{pmatrix} k. e_1^* \wedge e_2^* \otimes v = c.k.\det(A) \cdot e_1^* \wedge e_2^* \otimes v$. In particular, the action of G has orbits o and O = E\o and when D is a contact structure function of S takes values in O. if we consider the point $w = e_1^* \wedge e_2^* \otimes v \in$ E, the first reduction of the structure group traces out a unique 1-form η such that Ker (η) = D. It is easy to see that the second reduction defines an SO(2)- structures of coframes (η, η_1, η_2) that satisfy

$$\begin{cases} d\eta = \eta_1 \wedge \eta_2, & ; \\ g = \eta_1^2 + \eta_2^2, & , \end{cases}$$ Obviously, the operator φ that corresponds to dη is an almost complex structure on D and the metric g is associated with η.

The following example shows, when n ≥ 5 there might be no metrices associated with sub-Riemannian structure.

Example 3.4: Consider the space R^5 and a Sub-Riemannian structure (D, g), D = Ker $(dz + x_1 dy_1 + x_2 dy_2)$. $g = pdx_1^2 + qdy_1^2 + rdx_2^2 + sdy_2^2$. Denote by α $= dz + x_1 dy_1 + x_2 dy_2$, very contact 1-form η such that Ker (η) = D has a form f_α for some function f. Therefore the differential dη = df ∧ α + f(dx_1 ∧ dy_1 + dx_2 ∧ dy_2) when restricted to D is simply f(dx_1 ∧ dy_1 + dx_2 ∧ dy_2). Consider the following frame in D.

$$\begin{cases} e_1 = \partial/\partial x_1, & ; \\ e_2 = \partial/\partial x_2, & ; \\ e_3 = x_1 \partial/\partial z - \partial/\partial y_1 & ; \\ e_4 = x_2 \partial/\partial z - \partial/\partial y_2 & ; \end{cases}$$

With respect to this basis the operator that is adjoint to dη acts as follows.

$$\begin{cases} \varphi e_1 = \frac{f_1}{r} e_3, & ; \\ \varphi e_2 = \frac{f_1}{rs} e_4, & ; \\ \varphi e_3 = -\frac{f}{pe_1}, & ; \\ \varphi e_4 = -\frac{f}{qe_2}, & ; \end{cases}$$

and one may always find real numbers p, q, r, s such that g is not associated with η for any f.

4. FIRST REDUCTION:

We are going to solve the local equivalence problem by applying the Carten's Method of equivalence to k-structures of adopted coframes.

Statement of First Reduction: Let V be a (2n + 1)-dimensional vector space. Fix some basis $(e_1, e_2, \ldots, e_{2n}, v)$ in V. We want to calculate the A-image of Hom(V, g) in Hom(V ∧ V, V) and to find the orbit space $E = \frac{\text{Hom}(V \wedge V, V)}{A(\text{Hom}(V, g))}$.

Proposition 4.1: The space E may be identified with Hom $(V^1 \wedge V^1, V/V)$ where V^1 is a subspace in V generated by $e_1, e_2, \ldots, e_{2n}$.

Proof: Consider a set of matrices $\left\{ \begin{pmatrix} A & b \\ 0 & c \end{pmatrix} \mid A \in 0(2n, R), b \in R^{2n} \text{ and } c \in R \setminus \{0\} \right\}$.

Consider the following basis for g:

$$I_{pq}= e_p^* \otimes e_q - e_q^* \otimes e_p, \ p,q \in \overline{1, 2n}$$

$$II_k= v^* \otimes e_k, k \in \overline{1, 2n}$$

$$III= v^* \otimes v$$

The space Hom(V, g) is generated by vectors $e_s^* \otimes I_{pq}$, $e_s^* \otimes II_k$ and $e_s^* \otimes III$, where e_s is in $\{e_1, e_2, \ldots\ldots, e_{2n}, v\}$. Calculate the A-image of Hom(V, g) in Hom(V∧ V, V).

$$A(e_s^* \otimes II_k)(e_i, e_j) = e_s^*(e_i)II_k(e_j) -$$
$$e_s^*(e_j) II_k(e_i) =$$
$$e_s^*(e_i) v^*(e_j) e_k - e_s^*(e_j) v(e_j) e_k.$$ Therefore the image $e_s^* \otimes II_k$ in Hom(V∧ V, V) is a vector $(e_s^* \wedge v^*) \otimes e_k$. Analogously, $(e_s^* \otimes II_k) = (e_s^* \wedge v^*) \otimes v$. These vectors are zero in E. Therefore, in E we have that $(e_s^* \wedge e_q^*) \otimes e_p$. We want to show that for any indices i, j, k ∈(1, 2n), the image of $e_j^* \wedge e_k^* \otimes e_i$ is equal to zero in E. But we may write,

$$E_j^* \wedge e_k^* \otimes e_i = e_i \wedge e_k \otimes e_j = -e_k \wedge e_j \otimes e_i =$$
$$e_j \wedge e_k \otimes e_i = e_j \wedge e_i \otimes e_k.$$

This completes the proof.

Lemma 4.2: Two sub-Riemannian structures are locally equivalent if and only if their corresponding first reductions are locally equivalent.

Proof:- If the reduced structures are locally equivalent, then extending the structure group to C would give the equivalence of the original sub-Riemannian structures.

Conversely assume that S is locally equivalent to S. Denote by ϕ the equivalence an let c_1 and c_2 be the structure functions of S_∞ and $S_\in$. Since $c_1 o \phi^* = c_2$. We have that $c_1 o \phi^* \left(c_2^{-2}(s) \right) c_2 c_2^{-2}(s)$=s. This completes the proof.

5. SECOND REDUCTION:

Proposition 5.1: The orbit space E_1 of the K_1-structure S_∞ may be identified with the space A⊕ tHom(V'∧ V', V'/V') ⊕ Hom(V'∧ V / V', V / V'), where A is some subspace in Hom(V'∧ V', V').

Proof: The g_1 has the following basis:

$$II_k= v^* \otimes e_k, k= \overline{1, 2n}$$

$$A_{pq}= I_{pq} - J_0 I_{pq} - I_{qp} + J_0 I_{qp}, \ p,q= \overline{1, 2n}$$

Where J_0 is a standard complex structure of R^{2n}. Using the same arguments in Proposition 4.1, we may prove that the A-images of vectors $e_s^* \otimes II_k$ would be zero in E_1. These vectors would span the space Hom(V'∧ V / V', V'). The space Hom(V∧V,V) may be decomposed as direct sum.

Hom(V∧V,V) = Hom(V'∧ V', V / V') ⊕ Hom(V'∧ V / V', V') ⊕ Hom(V'∧ V', V') ⊕ Hom(V'∧ V / V', V / V'). Hence the proof.

Remark 5.2: Unless n =1 the subspace in Hom(V∧V, V) generated by the vectors $A(A_{pq})$ would not coincide with Hom(V'∧ V', V') and A would not be zero-dimensional.

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