Non-Trivial Positive Solutions for Some Classes of Non-Homogenous Non-Linear Diophantine Equations

Mathematics

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ABSTRACT *In this paper we will find a method to determine positive solutions for non-homogenous non-linear Diophantine equations of even degrees in several variables. Our main tool for this is the well-known Euler's theorem and the theory of congruence.*

INTRODUCTION

A Diophantine equation with m> 1 degree is a non-linear Diophantine equation. A Diophantine equation of degree m is called homogeneous non-linear if every term of the equation has total degree exactly m otherwise it is non-homogenous non-linear. For instance Pell's equation is the most famous class of non-homogenous non-linear Diophantine with the form $x^2 - Dy^2 = 1$ for some fixed positive integer D that is not a perfect square. Equations of higher degrees are harder to deal with.

By a solution of a Diophantine equation $p(x_1, x_2, x_3, \ldots, x_n)$ = 0 mean that an *n*-tuple of integers $x_1, x_2, x_3, \ldots, x_n$ such that the equation is satisfied. Euler phi-function φ is defined for each positive integer n by $\varphi(1) = 1$ and $\varphi(n)$ is the number of positive integers less than or equal to *n* that are co primes to n when $n > 1$. For instance $\varphi(2)$ =1, $\varphi(3)$=2, $\varphi(4)$=2, $\varphi(6) = 2$, $\varphi(7) = 6$. For a fixed positive integer n, integers a and b are said to be congruent modulo *n*, symbolized by $a \equiv b$ (mod *n*) if *n* divides the difference *a-b* otherwise $a \not\equiv b \pmod{n}$.

In this paper, we introduced method to determine non-trivial positive solutions for some class of *n*-degree non-homogenous non-linear Diophantine equation with $2^k + 1, k \geq 1$ possible number of variables and binomial coefficients and the number of these positive solutions by using Euler's theorem and the notion of congruence.

Theorem 1.1: If $p_1, p_2, p_3, \ldots, p_i$ are distinct prime factors of n, then $\varphi(n)$ =n $\prod_{j=1}^{i}(1 - \frac{1}{p_j})$.

Proof: Let p is a prime and n is a natural number. The natural numbers less than or equal to p^n are 1, 2, 3,…,p^i which are p^i in number. Among these Integers which are divisible by p are $p, 2p, 3p, \ldots, p(p^{i-1})$ which are p^{i-1} in number. Thus, the number of integers which are relatively prime to p^i is $p^i - p^{i-1}$=p^{i-1}(p-1). So, by the definition of Euler phi-function, $\varphi(p^i)$ =$p^i - p^{i-1} = p^{i-1}$(p-1).

Consider distinct prime factors $p_1, p_2, p_3, \ldots, p_i$ of n. By prime factorization n can be written as n=$p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_i^{\alpha_i}$. Then, since φ is multiplicative function and using Theorm 1.4

$\varphi(n) = \varphi(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_i^{\alpha_i})$= $\varphi(p_1^{\alpha_1})\varphi(p_2^{\alpha_2})\varphi(p_3^{\alpha_3}) \ldots \varphi(p_i^{\alpha_i})$

=$p_1^{\alpha_1-1}(p_1$-1)$p_2^{\alpha_2-1}(p_2$-1)$p_3^{\alpha_3-1}(p_3$-1)… $p_i^{\alpha_i-1}(p_i$-1)

$$= p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_i^{\alpha_i}(1 - \frac{1}{p_1}) (1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_i}) = \text{n} \prod_{j=1}^{i}(1 - \frac{1}{p_j}) .$$

Theorem1.2. $\varphi(n)$ is even positive integer which is less than n for n>2.

Proof: By definition $\varphi(n)$ is less than n for $n > 1$. From Theorem 1.6 and by considering the prime factorization of $n = p_1{}^{K_1} p_2{}^{K_2} p_3{}^{K_3} \ldots p_i{}^{k_i}$, for $k_1,\ k_2, \ldots, k_i \epsilon N$ and the primes $p_1, p_2, p_3, \ldots, p_n,$ the second part of this theorem is proved as follows:

$$\varphi(n) = n\frac{(p_1-1)(p_2-1)(p_3-1)\ldots(p_i)}{p_1 p_2 p_3 \ldots p_i}$$

$$\Rightarrow\ \varphi(n) = n\frac{(2\alpha_1)(2\alpha_2)(2\alpha_3)\ldots(2\alpha_i)}{p_1 p_2 p_3 \ldots p_i}\ ,\ \text{for } \alpha_1,\ \alpha_2, \ldots, \alpha_i \epsilon N$$

$$\Rightarrow \varphi(n) = 2m, \text{ for } m = 2^{i-1} \prod_{j=1}^{i} \alpha_j \prod_{j=1}^{i} p_j{}^{k_j-1} \in N.$$

Hence, $\varphi(n)$ is even positive integer less than n.

Corollary1.3. $\prod_{j=1}^{i}(p_j - 1)$ is even positive integer which divides $\varphi(n)$, for distinct prime factors $p_1, p_2, p_3, \ldots, p_{i,}$ of n.

Proof: From Theorem 1.1 and 1.2, We have a result that $D = \prod_{j=1}^{i} p_j{}^{k_j-1} = \frac{\varphi(n)}{\theta}$; for $n = \prod_{j=1}^{i} p_j{}^{k_j}$, $\theta = \prod_{j=1}^{i}(p_j - 1)$, $D \in N$ and since $\varphi(n)$ is even so is θ.

Algorithm for determining n for a given value of $\boldsymbol{\varphi(n)}$:

[1]Determine even divisors θ of $\varphi(n)$; for $n = \prod_{j=1}^{i} p_j{}^{k_j}$ and $\theta = \prod_{j=1}^{i}(p_j - 1)$ such that those identified θ are expressed as $\prod_{j=1}^{i}(p_j - 1)$.

[2]Determine the possible prime numbers $p_1,\ p_2, \ldots, p_i$ one by one and k_i corresponding to these p_i's using the property $\prod_{j=1}^{i} p_j{}^{k_j-1} = \frac{\varphi(n)}{\theta}$.

[3]Likewise, if there is no p_i obtained in step 2, then consider the possible p_i's as paire wise, triples up to n-tuples $p_1, p_2, \ldots, p_n$ and determine the corresponding k_i's.

[4]Using p_i's and k_i's which are identified in the above steps determine $n = \prod_{j=1}^{i} p_j{}^{k_j}$.

Theorem (Euler's) 1.4: If m is a positive integer and $(a, m) = 1$, then $a^{\varphi(m)} \equiv 1 (\mathrm{mod}\, m)$

Proof: let $\{r_1, r_2, r_3, \ldots, r_{\varphi(m)}\}$ be a completed reduced residue system (mod m). $\{ar_1, ar_2, ar_3, \ldots, ar_{\varphi(m)}\}$ is a complete reduced residue system (mod m). Hence each ar_i is congruent (mod m) to one and only one element of $\{r_1, r_2, r_3, \ldots, r_{\varphi(m)}\}$.
Thus, $ar_1. ar_2. ar_3 \ldots ar_{\varphi(m)} \equiv r_1. r_2. r_3 \ldots r_{\varphi(m)} (\mathrm{mod}\, m)$.

$$\Rightarrow \left(a^{\varphi(m)} - 1\right) r_1 r_2 r_3 \ldots r_{\varphi(m)} \equiv 0 (\mathrm{mod}\, m)$$

Therefore, m is a factor of $a^{\varphi(m)} - 1$, since $(r_1 r_2 r_3 \ldots r_{\varphi(m)}, m)=1$

Thus, $a^{\varphi(m)} \equiv 1 (mod\, m)$

Theorem 1.5: A non-linear Diophantine equation $x^n + y^n + z^n + w^n + \sum_{j=1}^{n-1} \binom{n}{j} [x^{n-j}y^j + z^{n-j}w^j] - r[(x+y)(z+w)] = 1$; for given even integers n has infinite non-trivial solution (a_1, a_2, b_1, b_2, c) with $(a_1 + a_2, b_1 + b_2, c) = 1$; $\varphi(a_1 + a_2) = n = \varphi(b_1 + b_2)$.

Proof: By Euler's theorem , $a^{\varphi(b)} \equiv 1 (mod\, b)$ and $b^{\varphi(a)} \equiv 1 (mod\, a)$. From this we have that $a^{\varphi(b)} - 1 \equiv 0 (mod\, b)$ and $b^{\varphi(a)} - 1 \equiv 0 (mod\, a)$.

Let D$=a^{\varphi(b)} + b^{\varphi(a)}$

Now, $D - 1 = a^{\varphi(b)} + (b^{\varphi(a)} - 1) \equiv 0\, mod\, a$, and $D \equiv 1 (mod\, b)$. Moreover; $D - 1 = (a^{\varphi(b)} - 1) + b^{\varphi(a)} \equiv 0\, mod\, b$, and $D \equiv 1 (mod\, b)$. Therefore, $D \equiv 1 (mod\, (ab))$.

$$\Rightarrow a^{\varphi(b)} + b^{\varphi(a)} \equiv 1 (mod(ab))$$

This congruence can be re-expressed as

$$a^{\varphi(b)} + b^{\varphi(a)} - (rab) = 1, \text{for some } r \epsilon\, Z. \qquad (1.1)$$

Now, put $a = a_1 + a_2$, $b = b_1 + b_2$ into the result put in Equation 1.1.

$$\text{It follows that } (a_1 + a_2)^n + (b_1 + b_2)^n - (a_1 + a_2)(b_1 + b_2)r = 1 . \qquad (1.2)$$

Using binomial expansion, Equation 1.2 becomes

$$\binom{n}{0} a_1^{\,n} + \binom{n}{1} a_1^{\,n-1}a_2^{\,1} + \cdots + \binom{n}{n-1} a_1^{\,1}a_2^{\,n-1} + \binom{n}{n} a_2^{\,n} + \binom{n}{0} b_1^{\,n} + \binom{n}{1} b_1^{\,n-1}b_2^{\,1} +$$
$$\binom{n}{2} b_1^{\,n-2}b_2^{\,2} + \cdots + \binom{n}{n-1} b_1^{\,1}b_2^{\,n-1} + \binom{n}{n} b_2^{\,n} - (a_1 + a_2)(b_1 + b_2)r = 1 \qquad (1.3)$$

By re-collecting similar terms and simplify coefficients Equation 1.3 becomes

$$a_1^{\,n} + a_2^{\,n} + b_1^{\,n} + b_2^{\,n} + \sum_{j=1}^{n-1} \binom{n}{j} [a_1^{\,n-j}a_2^{\,j} + b_1^{\,n-j}b_2^{\,j}] -$$

$$(a_1 + a_2)(b_1 + b_2)r = 1 \qquad (1.4)$$

Then, the result put in Equation 1.4 shows that $(x, y, z, w, r) = (a_1, a_2, b_1, b_2, c)$ can satisfy the non-homogenous non-linear Diophantine equation

$x^n + y^n + z^n + w^n + \sum_{j=1}^{n-1} \binom{n}{j} [x^{n-j}y^j + z^{n-j}w^j] - r[(x+y)(z+w)] =$ 1 for given even integers n; In such a way that c is calculated as $c = \frac{a^n + b^n - 1}{ab}$ where $a = a_1 + a_2; b = b_1 + b_2$. These two integers a and b can be computed based on the given positive even integers $\varphi(a) = \varphi(b) = n$ and using the algorithm given above.

Theorem 1.6: A non-linear Diophantine equation $k_1 x^n + k_2 y^n + k_3 z^n + k_4 w^n + \sum_{t=1}^{n-1} \binom{n}{t} [u_t x^{n-t}y^t + v_t z^{n-t}w^t] - r[(d_1 x + d_2 y)(d_3 z + d_4 w)] =$

1; for some positive even integer n and $k_1 = d_1{}^n$, $k_2 = d_2{}^n$, $k_3 = d_3{}^n$, $k_4 = d^n$, $u_t = d_1{}^{n-t}d_2{}^t$, $v_t = d_3{}^{n-t}d_4{}^t$ where $t = 1,2,3,\dots,n-1$ haspositive non-trivial solution with the form $(x_1 = x_0 - \frac{d_2}{(d_1,d_2)}d$, $y_1 = y_0 + \frac{d_1}{(d_1,d_2)}d$, $z_1 = z_0 - \frac{d_4}{(d_3,d_4)}e$, $w_1 = w_0 + \frac{d_4}{(d_3,d_4)}e$, $r = \frac{A^n+B^n-1}{AB})$ if

i. $-y_0\frac{(d_1,d_2)}{d_2} < d < x_0\frac{(d_1,d_2)}{d_1}$ and $-w_0\frac{(d_3,d_4)}{d_4} < e < z_0\frac{(d_3,d_4)}{d_3}$; where $d, e \in Z$ for particular solutions (x_0, y_0) and (z_0, w_0) of the Diophantine equations $d_1x + d_2y = A$ and $d_3z + d_4w = B$ respectively.

ii. $\varphi(A) = \varphi(B) = n$.

iii. $GCD\ (d_1, d_2)$ divides A and $GCD(d_3, d_4)$ divides B

Proof: To proof this Theorem, put $a = d_1a_1 + d_2a_2$, $b = d_3a_3 + d_4a_4$ into the resultput in Equation 1.1.It follows that

$$(d_1a_1 + d_2a_2)^n + (d_3a_3 + d_4a_4)^n - (d_1a_1 + d_2a_2)(d_3a_3 + d_4a_4)r = 1 . \qquad (1.5)$$

Using binomial expansion, Equation 1.2 becomes

$$\binom{n}{0}d_1{}^n a_1{}^n + \binom{n}{1}d_1{}^{n-1}d_2{}^1 a_1{}^{n-1}a_2{}^1 + \cdots + \binom{n}{n-1}d_1{}^1 d_2{}^{n-1}a_1{}^1 a_2{}^{n-1} + \binom{n}{n}d_2{}^n a_2{}^n +$$
$$\binom{n}{0}d_3{}^n a_3{}^n + \binom{n}{1}d_3{}^{n-1}d_4{}^1 a_3{}^{n-1}a_4{}^1 + \cdots + \binom{n}{n-1}d_3{}^1 d_4{}^{n-1}a_3{}^1 a_4{}^{n-1} + \binom{n}{n}d_4{}^n a_4{}^n -$$
$$(d_1a_1 + d_2a_2)(d_3a_3 + d_4a_4)r = 1 \quad (1.6)$$

By re-collecting similar terms and simplify coefficients Equation 1.6 becomes

$$d_1{}^n a_1{}^n + d_2{}^n a_2{}^n + d_3{}^n a_3{}^n + d_4{}^n a_4{}^n$$
$$+ \sum_{j=1}^{n-1}\binom{n}{j}d_1{}^{n-j}d_2{}^j a_1{}^{n-j}a_2{}^j + \sum_{j=1}^{n-1}\binom{n}{j}d_3{}^{n-j}d_4{}^j a_3{}^{n-j}a_4{}^j -$$
$$(d_1a_1 + d_2a_2)(d_3a_3 + d_4a_4)r = 1 (1.7)$$

Then, the result put in Equation 1.7 shows that $(x, y, z, w, r) = (x_1, y_1, z_1, w_1, c)$ can satisfy the non-homogenous non-linear Diophantine equation of the form:

$k_1x^n + k_2y^n + k_3z^n + k_4w^n + \sum_{t=1}^{n-1}\binom{n}{t}(d_1{}^{n-t}d_2{}^t)x^{n-t}y^t + \sum_{j=1}^{n-1}\binom{n}{j}(d_3{}^{n-j}d_4{}^j)z^{n-j}w^j$, $-r[(d_1x + d_2y)(d_3z + d_4w)] = 1$ where $k_1, k_2,\ k_3, k_4$ be defined by $k_1 = d_1{}^n$, $k_2 = d_2{}^n$, $k_3 = d_3{}^n$, $k_4 = d_4{}^n$ and c is calculated as $c = \frac{A^n+B^n-1}{AB}$ where $a = d_1a_1 + d_2a_2$; $b = d_3a_3 + d_4a_4$.

However; (x_1, y_1) and (z_1, w_1) are determined to be the general solutions of the linear Diophantine equations $d_1x + d_2y = A$ and $d_3w + d_4z = B$ with the property that GCD (d_1, d_2) divides A and $GCD\ (d_3, d_4)$ divides B respectively. And also, GCD $(d_1x_1 + d_2y_1$, $d_3z_1 + d_4w_1) = 1$, $\varphi(a = d_1x_1 + d_2y_1) = n = \varphi(b = d_3z_1 + d_4w_1)$.

These two integers a and b will be determined using the given positive even integers $n = \varphi(A) = \varphi(B)$, which is degree of the given non-linear Diophantine equation, and using the algorithm given above.

But by the notion of linear Diophantine equation, the general solutions of the two linear Diophantine equations $d_1x + d_2y = A$ and $d_3w + d_4z = B$ will bedetermined by $(x, y) = (x_1, y_1) = \left(x_0 - \frac{d_2}{(d_1,d_2)}d, y_0 - \frac{d_1}{(d_1,d_2)}d\right)$ and $(z, w) = (z_1, w_1) = (w_0 - \frac{d_4}{(d_3,d_4)}e, z_0 + \frac{d_4}{(d_3,d_4)}e)$ where $(x_0, y_0), (w_0, z_0)$ are particular solutions of the Diophantine equations $d_1x + d_2y = A$ and $d_3w + d_4z = B$, where $d, e \in Z$ are respectively.

Then we obtain d and e in such a way that this solution is positive as

$$x_1 = x_0 - \frac{d_2}{(d_1, d_2)}d > 0, y_1 = y_0 + \frac{d_1}{(d_1, d_2)}d > 0$$

$$z_1 = z_0 - \frac{d_4}{(d_3, d_4)}e > 0, w_1 = w_0 + \frac{d_4}{(d_3, d_4)}e > 0$$

$$\Rightarrow -y_0\frac{(d_1,d_2)}{d_2} < d < x_0\frac{(d_1,d_2)}{d_1} \text{ and } -w_0\frac{(d_3,d_4)}{d_4} < e < z_0\frac{(d_3,d_4)}{d_3}$$

Hence any such integers d and e satisfy these conditions give positive solutions.

Consequently; $(x, y, w, z, r) = (x_1, y_1, z_1, w_1, c) = (x_0 - \frac{d_2}{(d_1,d_2)}d, y_0 + \frac{d_1}{(d_1,d_2)}d, z_0 - \frac{d_4}{(d_3,d_4)}e, w_0 + \frac{d_4}{(d_3,d_4)}e, \frac{A^n+B^n-1}{AB})$ will be positive solutions for the non-linear Diophantine equation of the form $k_1x^n + k_2y^n + k_3z^n + k_4w^n + \sum_{j=1}^{n-1}\binom{n}{j}[(d_1^{n-j}d_2^{\,j})x^{n-j}y^j + (d_3^{n-j}d_4^{\,j})z^{n-j}w^j] - r[(d_1x + d_2y) + (d_3z + d_4w)] = 1$ where $k_1, k_2, \; k_3, k_4$ be defined by $k_1 = d_1^{\,n}, k_2 = d_2^{\,n}, k_3 = \; d_3^{\,n}, k_4 = d_4^{\,n}$ whenever $-y_0\frac{(d_1,d_2)}{d_2} < d < x_0\frac{(d_1,d_2)}{d_1}$ and $-w_0\frac{(d_3,d_4)}{d_4} < e < z_0\frac{(d_3,d_4)}{d_3}$.

Theorem1.7: The number of positive solutions to the Diophantine equation $k_1x^n + k_2y^n + k_3z^n + k_4w^n + \sum_{j=1}^{n-1}\binom{n}{j}[(d_1^{n-j}d_2^{\,j})x^{n-j}y^j + (d_3^{n-j}d_4^{\,j})z^{n-j}w^j] - r[(d_1x + d_2y) + (d_3z + d_4w)] = 1$ where $k_1 = d_1^{\,n}, k_2 = d_2^{\,n}, k_3 = \; d_3^{\,n}, k_4 = d_4^{\,n}$ is kl where k and l are the number of integers d and e respectively such that $-y_0\frac{(d_1,d_2)}{d_2} < d < x_0\frac{(d_1,d_2)}{d_1}$ and $-w_0\frac{(d_3,d_4)}{d_4} < e < z_0\frac{(d_3,d_4)}{d_3}$.

Proof: Let x_0, y_0, w_0, z_0 be particular solution to the given non-linear Diophantine equation $k_1x^n + k_2y^n + k_3z^n + k_4w^n + \sum_{t=1}^{n-1}\binom{n}{t}[u_tx^{n-t}y^t + v_tz^{n-t}w^t] - r[(a_1x + a_2y)(a_3z + a_4w)] = 1$, where $k_1 = d_1^{\,n}, k_2 = d_2^{\,n}, k_3 = \; d_3^{\,n}, k_4 = d_4^{\,n} \in Z^+$. From Theorem 1.13, we have that $x_1 = x_0 - \frac{d_2}{(d_1,d_2)}d, y_1 = y_0 + \frac{d_1}{(d_1,d_2)}d, z_1 = z_0 - \frac{d_4}{(d_3,d_4)}e, w_1 = w_0 + \frac{d_4}{(d_3,d_4)}e, r = \frac{A^n+B^n-1}{AB}$ is the general non-trivial solution for $d, e \in Z$.

And also, let k and l are the numbers of d and e respectively. Then, using the concepts of combination $\binom{k}{1}\binom{l}{1} = kl$ is the number of these positive solutions.

Theorem 1.8: $k_1 x_1{}^n + k_2 x_2{}^n + k_3 x_3{}^n + \cdots + k_4 x_{2h}{}^n + k_{2h+1} x_{2h+1}{}^n +$

$k_{2h+2} x_{2h+2}{}^n + \cdots + k_{4h} x_{4h}{}^n + \sum_{i=1}^{n-i} \binom{n}{i} [t_i x_1{}^{n-i} x_2{}^i + \cdots + u_i x_{2h-1}{}^{n-i} x_{2h}{}^i +$

$(k_1 x_1 + k_2 x_2)^{n-i} (k_3 x_3 + k_4 x_4)^i + \cdots + (k_{2h-3} x_{2h-3} +$

$k_{2h-2} x_{2h-2})^{n-i} (x_{2h-1} x_{2h-1} + x_{2h} x_{2h})^i + \cdots + (k_1 x_1 + k_2 x_2 + k_3 x_3 + \cdots +$

$k_h x_h)^{n-i} (k_{h+1} x_{h+1} + x_{h+2} x_{h+2} + k_{h+3} x_{h+3} + \cdots + k_{2h} x_{2h})^i] +$

$[(v_i x_{2h+1}{}^{n-i} x_{2h+2}{}^i + \cdots + p_i x_{4h-1}{}^{n-i} x^i{}_{4h}) + (k_{2h+1} x_{2h+1} +$

$k_{2h+2} x_{2h+2})^{n-i} (k_{2h+3} x_{2h+3} + k_{2h+4} x_{2h+4})^i + \cdots + (k_{4h-3} x_{4h-3} +$

$k_{4h-2} x_{4h-2})^{n-i} (k_{4h-1} x_{4h-1} + k_{4h} x_{4h})^i] + \cdots + (k_{2h} x_{2h+1} + k_{2h+2} x_{2h+2} +$

$k_{2h+3} x_{2h+3} + \cdots + k_{3h} x_{3h})^{n-i} (k_{3h+1} x_{3h+1} + k_{3h+2} x_{3h+2} + k_{3h+3} x_{3h+3} + \cdots +$

$k_{4h} x_{4h})^i - r(\sum_{i=1}^{2h} k_i x_i)(\sum_{j=1}^{2h} k_j x_{2h+j}) = 1$, for non-zero integer coefficients $k_i =$

$d_i{}^n, i = 1,2,3,\ldots,4h$ has infinite non-trivial positive Integer solution $(t_1, t_2, t_3, \ldots, t_{4i}, c)$ if

    i.   GCD$(d_1, d_2, d_3, \ldots, d_{2i})$ divides A and $(d_{2i+1}, d_{2i+2}, d_{2i+3}, \ldots, d_{4i})$ divides B.

    ii. $\varphi(A) = \varphi(B) = n$.

    iii.   the effective bound for $\varphi_i : -\theta_{2j} < \frac{(d_j, d_{j+1})}{d_{j+1}} < \varphi_i < \theta_{2j-1} \frac{(d_j, d_{j+1})}{d_j}$ in which $t_{2j-1} = \theta_{2j-1} - \frac{d_{2j}}{(d_{2j-1}, \, d_{2j})} \varphi_i, t_{2j} = \theta_{2j} - \frac{d_{j+1}}{(d_j, d_{j+1})} \varphi$ and the same $\varphi_i \in Z$ for the two consecutive integers $t_{2j-1}, t_{2j} \in Z^+$ for $j = 1,2,3,\ldots,2h$ and $i = 1,2,3,\ldots,2k$

Furthermore; this solution has the form $(t_1, t_2, t_3, \ldots, t_{4i}, c) = (\theta_1 - \frac{d_2}{(d_1, \, d_2)} \varphi, \theta_2 +$

$\frac{d_1}{(d_1, \, d_2)} \varphi, \theta_3 - \frac{d_4}{(d_3, \, d_4)} \varphi, \theta_4 + \frac{d_3}{(d_3, \, d_4)} \varphi, \ldots, \theta_{4h} - \frac{d_{4h}}{(d_{4h-1}, \, d_{4h})} \varphi, \theta_{4h} +$

$\frac{d_{4h-1}}{(d_{4j-1}, \, d_{4h})} \varphi, \frac{A^n + B^n - 1}{AB})$.

Proof: To proof this theorem the following linear combination of integers are considered:

$$a = a_1 + a_2; b = a_3 + a_4 \qquad (1^*)$$

$$a_1 = b_1 + b_2, \, a_2 = b_3 + b_4, a_3 = b_5 + b_6, a_4 = b_7 + b_8 \quad (2^*)$$

$b_1 = c_1 + c_2, b_2 = c_3 + c_4, b_3 = c_5 + c_6, \, b_4 = c_7 + c_8, b_5 = c_9 + c_{10}, b_6 = c_{11} + c_{12}, \, b_7 = c_{13} + c_{14}, b_8 = c_{15} + c_{16} (3^*)$

$$r_1 = d_1 t_1 + d_2 t_2, r_2 = d_3 t_3 + d_4 t_4, \ldots, r_{2k-1} = d_{2k-1} t_{2k-1} + d_{2k} t_{2k}, \ldots, r_{2k-1} =$$
$$d_{4k-1} t_{4k-1} + d_{4k} t_{4k} \qquad (4^*)$$

From 1*, 2*, 3* and 4* we get that

$$a = r_1 = d_1 t_1 + d_2 t_2 + d_3 t_3 + d_4 t_4 + \cdots + d_{2k} t_{2k} + d_{2k+} t_{2k} \text{ and}$$

$$b = d_{2k+1} t_{2k+1} + d_{2k+2} t_{2k+2} + \cdots + d_{4k+} t_{4k}.$$

(i). Put the partision $(1*)$ in to Equation 1.1. Then the result becomes

$$a_1{}^n + a_2{}^n + a_3{}^n + a_4{}^n + \sum_{j=1}^{n-1} \binom{n}{j} [a_1{}^{n-j}a_2{}^j + a_3{}^{n-j}a_4{}^j] -$$

$$(a_1 + a_2)(a_3 + a_4)r = 1 \tag{1.9}$$

(ii). Put the partision $(2*)$ in to Equation 1.9. Then the result becomes 2.0.

$$b_1{}^n + b_2{}^n + b_3{}^n + b_4{}^n + b_5{}^n + b_6{}^n + b_7{}^n + b_8{}^n + \sum_{j=1}^{n-1}\binom{n}{j} b_1{}^{n-j}b_2{}^j +$$

$$\sum_{j=1}^{n-1}\binom{n}{j} b_3{}^{n-j}b_4{}^j + \sum_{j=1}^{n-1}\binom{n}{j} b_5{}^{n-j}b_6{}^j + \sum_{j=1}^{n-1}\binom{n}{j} b_7{}^{n-j}a_8{}^j + \sum_{j=1}^{n-1}\binom{n}{j} (b_1 +$$

$$b_2)^{n-j}(b_3 + b_4)^j + \sum_{j=1}^{n-1}\binom{n}{j} (b_5 + b_6)^{n-j}(b_7 + b_8)^j - (b_1 + b_2 + b_3 + b_4)(b_5 +$$
$$b_6 + b_7 + b_8)r = 1 \tag{2.0}$$

Continuing in such a way we get that

(iii). Finally, by considering the partision $(4*)$. Then the result becomes 2.1

$$d_1{}^n t_1{}^n + d_1{}^n t_2{}^n + d_1{}^n t_3{}^n + \cdots + d_1{}^n t_{2k}{}^n + d_1{}^n t_{2k+1}{}^n + d_1{}^n t_{2k+2}{}^n + \cdots + d_1{}^n t_{4k}{}^n +$$
$$\sum_i^{n-i}\binom{n}{i}[(d_1{}^{n-i}d_2{}^i\, t_1{}^{n-i}t_2{}^i + \cdots + d_{2k-1}{}^{n-i}d_{2k}{}^i t_{2k-1}{}^{n-i}t_{2k}{}^i) + (d_1t_1 + d_2t_2)^{n-i}(d_3t_3 +$$
$$d_4t_4)^i + \cdots + (d_{2k-3}t_{2k-3} + d_{2k-2}t_{2k-2})^{n-i}(d_{2k-1}t_{2k-1} + d_{2k}t_{2k})^i + \cdots + (d_1t_1 +$$
$$d_2t_2+d_3t_3 + \cdots + d_{2k}t_{2k})^{n-i}(d_{2k+1}t_{2k+1} + d_{2k+2}t_{2k+2} + d_{2k+3}t_{2k+3} + \cdots + d_{2k}t_{2k})^i +$$
$$(d_{2k+1}{}^{n-i}d_{2k+2}{}^i t_{2k+1}{}^{n-i}t_{2k+2}{}^i + \cdots + d_{4k-1}{}^{n-i}d^i{}_{4k}t_{4k-1}{}^{n-i}t^i{}_{4k}) + (d_{2k+1}t_{2k+1} +$$
$$d_{2k+2}t_{2k+2})^{n-i}(d_{2k+3}t_{2k+3} + d_{2k+4}t_{2k+4})^i + \cdots + (d_{4k-3}t_{4k-3} +$$
$$d_{4k-2}t_{4k-2})^{n-i}(d_{4k-1}t_{4k-1} + d_{4k}t_{4k})^i + \cdots + (d_{2k+1}t_{2k+1} + d_{2k+2}t_{2k+2} + d_{2k+3}t_{2k+3} + \cdots +$$
$$d_{3k}t_{3k})^{n-i}(d_{3k+1}t_{3k+1} + d_{3k+2}t_{3k+2} + d_{3k+3}t_{3k+3} + \cdots + d_{3k+3}t_{4k})^i] -$$
$$r\left(\sum_{i=1}^{2k} d_it_i\right)\left(\sum_{j=1}^{2k} d_{2k+j}t_{2k+j}\right) = 1. \tag{2.1}$$

Finally by continuous substitution as above in (i) and (ii) a and b are expressed as follows:

$$a = \sum_{i=1}^{2k} d_it_i = d_1t_1 + d_2t_2 + d_3t_3 + \cdots + d_{2k}t_{2k} \tag{2.2}$$

$$b = \sum_{j=1}^{2k} d_{2k+j}t_{2k+j} = d_{2k+1}t_{2k+1} + d_{2k+2}t_{2k+2} + d_{2k+3}t_{2k+3} + \cdots + d_{2k}t_{2k} \tag{2.3}$$

Such that $\varphi(a) = n = \varphi(b) = n$ where $a, b \in Z^+$.

Since by the assumption $GCD(d_1, d_2, d_3, \dots, d_{2k})$ divides A and $(d_{2i+1}, d_{2i+2}, d_{2i+3}, \dots, d_{4k})$ divides B, Equation 2.2 and 2.3 show that $a = d_1x_1 + d_2x_2 + d_3x_3 + \cdots + d_{2k}x_{2k}$ and $b = d_{2k+1}x_{2k+1} + d_{2k+2}x_{2k+2} + d_{2k+3}x_{2k+3} + \cdots + d_{4k}x_{4k}$ has infinite non $-$ trivial soltuion $(t_1, t_2, t_3, \dots, t_{2k})$ and $(t_{2k+1}, t_{2k+2}, t_{2k+3}, \dots, t_{4k})$ respectively as they are consistent and dependent.

Consequently; a result in Equation 2.1 shows that the non-linear Diophantine Equation 1.8 in expressed in this theorem 1.8 has infinite non $-$ trivial solution $(x_1, x_2, x_3, \dots, x_{2k}, x_{2k+1}, x_{2k+2}, x_{2k+3}, \dots, x_{4k})$ $=(t_1, t_2, t_3, \dots, t_{2k}, t_{2k+1}, t_{2k+2}, t_{2k+3}, \dots, t_{4k}, r)$ where $\varphi(a) = \varphi(t_1 + t_2 + t_3 + \cdots + t_{2k}) = \varphi(b) = \varphi(t_{2k+1} + t_{2k+2} + t_{2k+3} + \cdots + t_{4k}) = n$.

Theorem1.9: The number of positive solutions of the Diophantine equation in Theorem 1.8 is equal to $N = \prod_{j=1}^{2k} n_j$, for n_j equal to the cardinality of the set $\{\varphi_i \in Z: -\theta_{2j} < \frac{(d_j, d_{j+1})}{d_{j+1}} < \varphi_i < \theta_{2j-1} \frac{(d_j, d_{j+1})}{d_j}\}$, where $i=1,2,3,...,2k$ and $j=1,2,3,...,2k$.

Proof: The Diophantine equation of the form which is put in Theorem 1.15 will have positive solutions with the effective bounds of φ_i as $-\theta_{2j} < \frac{(d_j, d_{j+1})}{d_{j+1}} < \varphi_i < \theta_{2j-1} \frac{(d_j, d_{j+1})}{d_j}$.

Now, let $S_i = \{\varphi_i \in Z: -\theta_{2j} < \frac{(d_j, d_{j+1})}{d_{j+1}} < \varphi_i < \theta_{2j-1} \frac{(d_j, d_{j+1})}{d_j}\}$.

The number of φ_i is equal to the cardinality of S_i, denoted as $|S_i|$. Since any two consecutive variables will have the same number of positive solutions, the Equation put in Theorem 1.8 will have 2j number of positive solutions for $j=1,2,3,...,2h$. So starting from the first two pair wise consecutively up to the $(2j)^{th}$ variable the possible number of positive solutions will be $\binom{n_1}{1}\binom{n_2}{1}\binom{n_3}{1}...\binom{n_{2k}}{1} = n_1 n_2 n_3 ... n_{2k} = \prod_{j=1}^{2k} n_j$.

Example 1: For $\varphi(n) = 4$ the possible values of n corresponding to the value 4 are obtained as follows using the above series of steps:

Step1: The possible even divisors $\varphi(n) = 4$ are $\theta = 1, 2, 4$.

Step2: $i.$ For $\theta = 1 = \prod_{j=1}^{i}(p_j - 1)$, for satisfying this equality there is no prime numbers except $p_1 = 2$. So $\frac{\varphi(n)}{\theta} = \frac{4}{1} = 4 = \prod_{j=1}^{i} p_j^{k_j - 1}$. Hence, the possible k_1 corresponding to $p_1 = 2$ is obtained as $p_1^{k_1 - 1} = 2^{k_1 - 1} = 4$. Therefore; $k_1 = 3$. Finally, $n = \prod_{j=1}^{i} p_j^{k_j} = 2^{k_1} = 8$.

$ii.$ For $\theta = 2 = \prod_{j=1}^{i}(p_j - 1)$, the prime numbers corresponding to this θ are:

  a. $p_1 = 3$. So by the property $\prod_{j=1}^{i} p_j^{k_j - 1} = \frac{\varphi(n)}{\theta}$ we have that $3^{k_1 - 1} = 2$. From this it is impossible to get an integer k_1 satisfying this equality.

  b. $p_1 = 2$ and $p_2 = 3$ Pairwise. So by the property $\prod_{j=1}^{i} p_j^{k_j - 1} = \frac{\varphi(n)}{\theta}$ we have that $2^{k_1 - 1}. 3^{k_2 - 1} = 2$. As a result; $k_1 = 2$ and $k_2 = 1$. Finally, $n = \prod_{j=1}^{i} p_j^{k_j} = 2^{k_1}. 3^{k_2} = 12$.

$iii.$ For $\theta = 4 = \prod_{j=1}^{i}(p_j - 1)$, the prime numbers corresponding to this θ are:

  a. $p_1 = 5$. So by the property $\prod_{j=1}^{i} p_j^{k_j - 1} = \frac{\varphi(n)}{\theta}$ we have that $5^{k_1 - 1} = 1$. From this $k_1 = 1$. Therefore; $n = \prod_{j=1}^{i} p_j^{k_j} = 5^{k_1} = 5$.

  b. $p_1 = 2$ and $p_2 = 5$ Pairwise. So by the property $\prod_{j=1}^{i} p_j^{k_j - 1} = \frac{\varphi(n)}{\theta}$ we have that $2^{k_1 - 1}. 5^{k_2 - 1} = 1$. As a result; $k_1 = 1$ and $k_2 = 1$. Finally, $n = \prod_{j=1}^{i} p_j^{k_j} = 2^{k_1}. 5^{k_2} = 10$. Hence; the values of n corresponding to $\varphi(n) = 4$ are 5,8,10,12.

Example2: Some of the nontrivial solutions for $x^4 + y^4 + z^4 + w^4 + \sum_{t=1}^{3} \binom{4}{t} [x^{n-t}y^t + z^{n-t}w^t] = (x+y)(z+w)r + 1$ *are (t, r, s, d, k)=*
(2,3,1,7,86),(2,3,2,6,86),(2,3,6,2,86),(2,3,3,5,86),(2,3,5,3,86),(2,3,4,4,86),(1,4,1,7,86),(1,4,7,1,86),(1,4,2,6,86),(1,4,6,2,86),(1,4,3,5,86),(1,4,4,4,86).

Example 3: $x^4 + 16y^4 + 81w^4 + z^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 27w^3z + 9w^2z^2 + 3wz^3 - (x+2y)(3w+z)r$=1 will have solution (x,y,z,w,r)=(1-2t,2+t,2-k,2+3k,118) where (1,2) and (2,2) are particular solutions of $x + 2y = 5$ and $3w + z = 8$ in such a way that $\varphi(5) = \varphi(8) = 4$, which is degree of the equation, and r=$\frac{8^4+5^4-1}{8x5}$ = 118.

Example 4: Some of the nontrivial positive solutions for the non-linear Diophantine equation $x^8 + y^8 + z^4 + w^4 + \sum_{t=1}^{7}\binom{8}{t}x^{8-t}y^t + \sum_{t=1}^{3}\binom{4}{t}z^{4-t}w^t - r(x+y)(z+w)$=1 are:

$(5,10,4,4,\frac{15^8+8^4-1}{120},)$, $(12,3,3,5,\frac{15^8+8^4-1}{120})$, $(11,4,2,6,\frac{15^8+8^4-1}{120})$, $(9,6,2,6,\frac{15^8+8^4-1}{120})$, $(7,8,1,7,\frac{15^8+8^4-1}{120})$, $(13,3,4,1,\frac{15^8+8^4-1}{120})$, $(10,6,3,2,\frac{15^8+8^4-1}{120})$, $(8,8,4,1,\frac{15^8+8^4-1}{120})$, $(7,9,4,1,\frac{15^8+8^4-1}{120})$, $(12,12,4,1,\frac{15^8+8^4-1}{120})$, $(20,4,3,2,\frac{15^8+8^4-1}{120})$, $(18,6,3,2,\frac{15^8+8^4-1}{120})$, $(17,7,1,4,\frac{15^8+8^4-1}{120})$, $(13,11,3,2,\frac{15^8+8^4-1}{120})$, $(10,14,1,4,\frac{15^8+8^4-1}{120})$. As $\varphi(b)= \varphi(15) = 8$; $\varphi(8) = 4$ and $c = \frac{a^n+b^n-1}{ab} = \frac{15^8+8^4-1}{120}$ using Theorem 1.8

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