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APPROXIMATION OF COMMON FIXED POINTS OF THREE QUASI-CONTRACTION MAPPINGS

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ABSTRACT

In this paper, we prove a convergence theorem with error estimates for the Jungck iteration for approximating point of coincidence and common fixed points of three quasi-contraction type mappings in cone metric spaces. The result is new even in the classical metric setting.

1. INTRODUCTION

In 1971, Ćirić [3] proved an interesting generalization of the Banach contraction principle. A self-mapping T of a metric space (X,d) is called a generalized contraction if

$$d(Tx,Ty) \leq \lambda \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

for all $x, y \in X$, where $\lambda \in [0,1)$ is a constant.

Theorem 1.1 [3]. Let T be a generalized contraction mapping of a complete metric space (X,d) with contraction constant $\lambda \in [0,1)$. Then:

(i) T has a unique fixed point ξ in X.

(ii) For every $x_0 \in X$ the Picard iteration $(x_n)_{n=0}^{\infty}$, defined by $x_{n+1} = Tx_n$, converges to ξ.

(iii) For every $n \geq 0$ the error estimate holds

$$d(x_n,\xi) \preceq \frac{\lambda^n}{1-\lambda} d(x_0,x_1).$$

The part (i) of Theorem 1.1 was generalized for common fixed points of two mappings in cone metric spaces by Azam and Rhoades [2], Ding, Jovanović, Kadelburg and Radenović [5] and Jungck, Radenović, Radojević and Rakočević [10]. Also part (i) was generalized for common fixed points of three mappings in cone metric spaces by Olaleru [12]. Very recently Proinov and Nikolova [15] obtained a complete generalization of Theorem 1.1 with error estimates for Jungck iteration for approximating common fixed points of three mappings in cone metric spaces.

In 1974, Ćirić [4] introduced the class of the so-called quasi-contractions mappings. A self-mapping X of a metric space (X,d) is called a quasi-contraction if

$$d(Tx,Ty) \leq \lambda \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for all $x, y \in X$, where $\lambda \in [0,1)$ is a constant. The following well-known fixed point theorem of Ćirić improves Theorem 1.1.

Theorem 1.2 [4]. For every quasi-contraction mapping T of a complete metric space (X,d) with contraction constant $\lambda \in [0,1)$, the conclusions (i), (ii), (iii) of Theorem 1.1 hold true.

The parts (i) and (ii) of Theorem 1.2 were generalized for fixed points of quasi-contraction mappings in cone metric spaces by Ding, Jovanović,

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Kadelburg and Radenović [5], Gajić and Rakočević [6], Ilić and V. Rakočević [7] and Zhang [17]. Recently, Kadelburg, Radenović and Rakočević [11] generalized part of Theorem 1.2 for two common fixed points of two quasi-contraction type mappings in cone metric spaces. Very recently Proinov and Nikolova [16] obtained a complete generalization of Theorem 1.2 with error estimates for Jungck iteration for approximating common fixed points of two quasi-contraction type mappings in cone metric spaces.

In this paper we shall prove a convergence theorem with error estimates for Jungck iteration for approximating common fixed points of three quasi-contraction type mappings provided that $\lambda \le 1/2$. The problem in the case $\lambda > 1/2$ is open. We state our result in cone metric space but it is new even in the classical metric space.

2. MAIN RESULT

Let f and g be two self-mappings of a nonempty set X. Recall that a point $x \in X$ is called a *coincidence point* of f and g if $fx = gx$. A point $y \in X$ is called a *point of coincidence* of f and g if there exists a point $x \in X$ such that $y = fx = gx$. Mappings f and g are called *weakly compatible* [9] if they commute at their coincidence points.

Definition 2.1 [17]. Let $(Y, \preceq)$ be an ordered vector space, $x \in Y$ and $A \subset Y$. Then we say that

$$x \preceq A$$

if there exists a vector $y \in A$ such that $x \preceq y$.

Definition 2.2 (Jungck iteration [8]). Let T, S and f be self-mappings of a nonempty set X and let $(x_n)_{n=0}^{\infty}$ be a sequence in X such that

$$f x_{n+1} = \begin{cases} Tx_n & if \quad n \quad is\ even, \\ Sx_n & if \quad n \quad is\ odd, \end{cases} \quad n = 0,1,2,...$$

Then the sequence $(f x_n)_{n=0}^{\infty}$ is said to be a (T,S)-f-*sequence* or *Jungck iteration*.

For brevity, throughout the paper we denote by $\mathrm{M}(T,S,f,x,y)$ the set $\{d(fx,fy), d(fx,Tx), d(fy,Sy), d(fx,Sy), d(fy,Tx)\}$, where T,S,f are self-mappings of a cone metric space X and $x,y \in X$.

Throughout the paper $\mathrm{co}\, A$ stands for the convex hull of a set A in a real vector space Y.

Now we can state the main result of this paper.

Theorem 2.3. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let T,S,f be self-mappings of X such that

$$T(X) \cup S(X) \subset f(X).$$

Suppose the following conditions hold:

(a) $d(Tx,Ty) \preceq \lambda\, \mathrm{co}\, \mathrm{M}(T,S,f,x,y)$ for all $x,y \in X$ with $x \ne y$, where $\lambda \in [0,1/2)$ is a constant.

(b) $d(Tx,Sx) \ne d(fx,Tx) + d(fx,Sx)$ for every $x \in X$ whenever $Tx \ne Sx$.

(c) $f(X)$ or $T(X) \cup S(X)$ is a complete sub-space of X.

Then the following statements hold true:

(i) The mappings T,S,f have a unique point of coincidence $\xi \in X$.

(ii) Every (T,S)-f-sequence $(f x_n)$ in the space X converges to ξ.

(iii) For every $n \ge 0$ the following a priori error estimate holds

$$d(fx_n,\xi) \preceq \frac{\theta^n}{1-\theta} d(fx_0,Tx_0),$$

where $\theta = \lambda/(1-\lambda)$.

(iv) For every $n \ge 0$ the following two a posteriori error estimates hold:

$$d(fx_n,\xi) \preceq \frac{1}{1-\theta} d(fx_n, fx_{n+1})$$

$$d(fx_{n+1},\xi) \preceq \frac{\theta}{1-\theta} d(fx_{n+1}, fx_n).$$

(v) If the pairs (T,f) and (S,f) are weakly compatible, then ξ is a unique common fixed point of T,S,f.

Remark 2.2. If condition (a) of Theorem 2.3 is satisfied for all $x,y \in X$, then assumption (b) of this theorem can be dropped.

3. AUXILIARY LEMMAS

In this section, we give some lemmas which will be used in the proof of the main result.

Lemma 3.1 [14]. Let $(Y,\preceq)$ be an ordered vector space. Suppose that $x,y,x_1,...,x_n,y_1,...,y_m$ are nonnegative vectors in Y and $0 \le \lambda < 1$. Then:

(i) $x \preceq \text{co}\{x_1,...,x_n,y\}$, $y \preceq \text{co}\{y_1,...,y_m\} \Rightarrow x \preceq \text{co}\{x_1,...,x_n,y_1,...,y_m\}$;

(ii) $x \preceq \text{co}\{\lambda x,x_1,...,x_n\} \Leftrightarrow x \preceq \text{co}\{x_1,...,x_n\}$;

(iii) $x \preceq \text{co}\{x_1,...,x_n,y\} \Leftrightarrow x \preceq \text{co}\{x_1,...,x_n\}$ if $y = x_i$ for some i.

Lemma 3.2 [13]. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let (x_n) be an infinite sequence in X satisfying

$d(x_{n+1},x_{n+2}) \preceq \lambda\, d(x_n,x_{n+1})$ for every $n \ge 0$,

where $\lambda \in [0,1)$ is a constant. Then (x_n) is a Cauchy sequence. Moreover if (x_n) converges to a point $\xi \in X$, then for all $n \ge 0$ the following error estimates hold:

$$d(x_n,\xi) \preceq \frac{\lambda^n}{1-\lambda}\, d(x_0,x_1);$$

$$d(x_n,\xi) \preceq \frac{1}{1-\lambda}\, d(x_n,x_{n+1});$$

$$d(x_{n+1},\xi) \preceq \frac{\lambda}{1-\lambda}\, d(x_{n+1},x_n).$$

Lemma 3.3 [16]. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let T,S and f be self-mappings of X satisfying conditions

(a) $d(fx,Tx) \preceq a\, d(fx,fy) + b\, d(fx,Sy)$ for all $x,y \in X$ with $x \ne y$,

(b) $d(fx,Sx) \preceq a\, d(fx,fy) + b\, d(fx,Ty)$ for all $x,y \in X$ with $x \ne y$,

(c) $d(Tx,Sx) \ne d(fx,Tx) + d(fx,Sx)$ for every point $x \in X$ whenever $Tx \ne Sx$,

where a and b are nonnegative constants. If there exists a (T,S)-f-sequence in X converging to a point $\xi \in f(X)$, then ξ is a point of coincidence of T,S,f.

The next two lemmas were stated in [15] without proofs. For the sake of completeness we present their proofs.

Lemma 3.4 [15]. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let T,S,f be self-mappings of X. Suppose that there exists $\lambda \in [0,1)$ such that for all $x,y \in X$ with $x \ne y$,

$$d(Tx,Sy) \preceq \lambda\, \text{co}\, M\{T,S,f,x,y\}.$$

Then for all $x,y \in X$ with $x \ne y$ the following inequalities hold::

$$d(fx,Tx) \preceq a\, d(fx,fy) + b\, d(fx,Sy),$$

$$d(fx,Sx) \preceq a\, d(fx,fy) + b\, d(fx,Ty),$$

where $a = \lambda/(1-\lambda)$ and $b = (1+\lambda)/(1-\lambda)$.

Proof. Let $x,y \in X$ be such that $x \ne y$. It follows from the assumption of the lemma that there exist five nonnegative numbers $\alpha,\beta,\gamma,\mu,\upsilon$ such that $\alpha+\beta+\gamma+\mu+\upsilon = 1$ and

$$d(Tx,Sy) \preceq \lambda\,[\alpha\, d(fx,fy) + \beta\, d(fx,Tx) + \gamma\, d(fy,Sy) + \mu\, d(fx,Sx) + \upsilon\, d(fy,Tx)].$$

From this and the triangle inequality, we easily obtain

$$d(Tx,Sy) \preceq \lambda\,[d(fx,fy) + d(fx,Sy) + d(fx,Tx)].$$

Again from the triangle inequality, we get

$$d(fx,Tx) \preceq d(fx,Sy) + d(Tx,Sy)$$

Combining the last two inequalities, we obtain

$$(1-\lambda)\, d(fx,Tx) \preceq$$
$$\lambda\, d(fx,fy) + (1+\lambda)\, d(fx,Sy)$$

which yields the first desired inequality. The second inequality can be obtained similarly. □

Lemma 3.5 [15]. Under the assumptions of Lemma 3.4, the mappings T,S,f have at most one point of coincidence in X.

Proof. Assume that u and v are two distinct point of coincidence of T,S,f, that is

$$u = Tx = Sx = fx \text{ and } v = Ty = Sy = fy$$

for some $x, y \in X$ with $x \neq y$. Then taking into account the property (iii) of Lemma 3.1, we can write the inequality

$$d(Tx,Sy) \preceq \lambda \operatorname{co} M\{T,S,f,x,y\}$$

in the form

$$d(u,v) \preceq \lambda\, d(u,v).$$

This yields $u = v$ since $\lambda < 1$. This contradiction proves our claim. □

Lemma 3.6 [1]. Let X be a nonempty set and T,S,f be self-mappings of X that have a unique point of coincidence $\xi \in X$. If (T,f) and (S,f) are weakly compatible, then ξ is a unique common fixed point of T,S,f.

4. PROOF OF THE MAIN RESULT

First we shall prove that every (T,S)-f-sequence (fx_n) satisfies the following inequality

(1) $$d(fx_{n+1}, fx_{n+2}) \preceq \theta\, d(fx_n, fx_{n+1}),$$

for every $n \geq 0$.

If $x_n = x_{n+1}$ for some $n \geq 0$, then to prove the inequality (1) it is sufficient to prove

(2) $$fx_{n+2} = fx_{n+1}.$$

Assume that $fx_{n+2} \neq fx_{n+1}$. If n is an even number, then $fx_{n+2} = Sx_{n+1} = Sx_n$ and $fx_{n+1} = Tx_n$ which implies $Tx_n \neq Sx_n$. Then it follows from condition (b) that

$$d(Tx_n,Sx_n) \neq d(fx_n,Tx_n) + d(fx_n,Sx_n).$$

Because of $fx_n = fx_{n+1} = Tx_n$, the last inequality becomes $d(Tx_n,Sx_n) \neq d(Tx_n,Sx_n)$ which is a contradiction. This proves (2) when n is an even number. The case when n is odd can be proved analogously.

Now suppose that $x_n \neq x_{n+1}$ for some $n \geq 0$. If n is an even number, then

$$d(fx_{n+1}, fx_{n+2}) = d(Tx_n, Sx_{n+1}).$$

Applying condition (a) to the right-hand side of this equality and taking into account claims (ii) and (iii) of Lemma 3.1, we obtain

(3) $$d(fx_{n+1}, fx_{n+2}) \preceq$$
$$\lambda \operatorname{co}\left\{d(fx_n, fx_{n+1}), d(fx_n, fx_{n+2})\right\}.$$

Similarly, if n is an odd number, then

$$d(fx_{n+1}, fx_{n+2}) = d(Sx_n, Tx_{n+1}).$$

From condition (a) and properties (ii) and (iii) of Lemma 3.1, we again get (3). It follows from (3) that there exists $\alpha \in [0,1]$ such that

(4) $$d(fx_{n+1`}, fx_{n+2}) \preceq$$
$$\lambda\, [\alpha\, d(fx_n, fx_{n+1}) + (1-\alpha)\, d(fx_n, fx_{n+2})],$$

By the triangle inequality we have

$$d(fx_n, fx_{n+2}) \preceq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}).$$

From this and (4), we get

$$d(fx_{n+1}, fx_{n+2}) \preceq$$
$$\lambda\, d(fx_n, fx_{n+1}) + \lambda\, d(fx_{n+1}, fx_{n+2})$$

which implies (1).

It follows from (1) and Lemma 3.2 that (fx_n) is a Cauchy sequence. Then it follows from condition (c) that (fx_n) converges to a point $\xi \in f(X)$. This completes the proof of (ii). The error estimates (iii), (iv) and (v) follow immediately from Lemma 3.2.

From Lemma 3.4 and the fact that the sequence (fx_n) converges the point $\xi \in f(X)$

follows that all assumptions of Lemma 3.3 are satisfied. Now it follows from Lemma 3.3 that ξ is a point of coincidence of the mappings T, S, f. From Lemma 3.5 we conclude that ξ is unique. This proves (i).

If the pairs (T, f) and (S, f) are weakly compatible, then from Lemma 3.6 follows that ξ is a unique fixed point of T, S, f. This proves (vi) which completes the proof. □

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