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Quadratic Type Minimum Risk Equivariant (MRE) estimation of Uniform Location-Scale Model

Statistics

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ABSTRACT

In this paper, by assuming that a progressive Type II right censored sample is available, we obtain QA-MRE estimators for the vector parameter of $(\xi, \tau)'$ based on the Type II progressive right censored sample. Further MRE estimator of the vector parameter $(\xi, \tau)'$ is obtained with respect toLinex type loss function. The paper is organized as follows: Section 2 deals with the problem of QA-MRE estimator for the vector parameter based on Quadratic type loss function and Minimum Risk Equivariant(MRE) estimation of the parameter under Linex loss function (Varian,1975).

1.Introduction

Equivariance is a desirable property used for restricting the class of estimators whenever the model possesses symmetry. ZACKS (1971) and LEHMANN AND CASELLA (1998) provide a detailed study of the problem of equivariant estimation for certain models. In the case of location-scale model, LEHMANN AND CASELLA (1998) develops marginal Equivariant procedure for estimating the parameters. EDWIN PRABAKARAN and CHANDRASEKAR (1994) have proposed a simultaneous Equivariant estimation for estimating the parameters of a location-scale model. For a detailed discussion on simultaneous Equivariant estimation and related results the reader is referred to EDWIN PRABAKARAN (1995).Contributions to simultaneous Equivariant estimation based on censored samples studied in Leo Alexander(2000).

1.1 Preliminaries

Let N denote the total number randomly selected items put to test simultaneously and n designate the number of samples specimens which fail. Thus the number of completely determined life spans is n. At the time of the i-th failure, r_i surviving units are randomly withdrawn from the test, i=1,2,…,n. Clearly, $r_n = N - n - \sum_{i=1}^{n-1} r_i$.

Let $X_{i:N}, i = 1,2,\ldots,n$, denote the failure times of the completely observed times. Then, the joint probability density function (pdf) of $(X_{1:N}, X_{2:N},\ldots, X_{n:N})$ is

$$g_\theta(x_1, x_2,\ldots,x_n) = \prod_{i=1}^{n}(N - \sum_{j=1}^{i-1} r_j - i + 1)\; f_\theta(x_i)\{1 - F_\theta(x_i)\}^{r_i} \;. \quad \ldots(1.1)$$

Here, f_θ *and* F_θ denote the common pdf and distribution function of the N items under life-test. Further, $r_1, r_2,\ldots, r_n$ are assumed to be pre-fixed by the experimenter.

1. Location –Scale Model

In this case, the pdf is taken to be

$$f_\theta(x) = \begin{cases} 1/\tau \;, & \xi \le x \le \xi + \tau \;;\; \xi \in R, \tau > 0 \\ 0 \;, & otherwise \end{cases}$$

`

Note that $\theta = (\xi, \tau)'$. Thus (1.1) reduces to

$$g_\theta(x_1,\ldots,x_n) = \{\prod_{i=1}^{n}(N - \sum_{j=1}^{i-1} r_j - i + 1)\;\}\,(1/\tau^n)\;\prod_{i=1}^{n}\{1 - (x_i - \xi)/\tau$$

$$\xi \le x_1 \le x_n < \xi + \tau ; \xi \in R , \tau > 0. \qquad \ldots(2.1)$$

Thus the joint distribution of $(X_{1:N}, X_{2:N},\ldots, X_{n:N})$ belongs to a location – scale family with location-scale parameter $\theta = (\xi, \tau)'$. We are interested in finding the

$Q_A - MRE$ estimator for vector parameter $(\xi,\tau)'$ and the MRE estimator for the vector parameter $(\xi,\tau)'$ based on Linex loss function. By simultaneous equivariant estimation approach of Edwin Prabakaran and Chandrasekar (1994), we obtain the $Q_A - MRE$ estimator of $(\xi,\tau)'$.

Case (i): Assume that the loss function is of the quadratic type (Zacks 1971,p.102), let us consider the problem of estimating $(\xi,\tau)'$. In order to obtain the MRE estimator of $(\xi,\tau)'$, take $\delta_{01}(\mathbf{X}) = X_{1:N}$ and $\delta_{02}(\mathbf{X}) = X_{n:N} - X_{1:N}$.

Clearly $\delta_0(\mathbf{X}) = (\delta_{01}(\mathbf{X}), \delta_{02}(\mathbf{X}))'$ is a location – scale equivariant estimator and also $(X_{1:N}, X_{n:N})'$ is a sufficient statistics but not complete. In order to evaluate $(w_1^*, w_2^*)'$, where

$$w_1^* = [a_{11}a_{22}E(\delta_{02}^2 \mid z)E(\delta_{01}g \mid z) - a_{12}^2 E(\delta_{01}g \mid z)E(\delta_{01}\delta_{02} \mid z) - a_{12}a_{22}\{E(\delta_{02}^2 \mid z)E(g \mid z) - E(\delta_{02}g \mid z)E(\delta_{02} \mid z)\}] / \{a_{11}a_{22}E(g^2 \mid z)E(\delta_{02}^2 \mid z) - a_{12}^2 E^2(\delta_{02}g \mid z\} \quad ...(2.2)$$

and

$$1/w_2^* = [-a_{11}a_{12}\{E(g^2 \mid z)E(\delta_{01}\delta_{02} \mid z) - E(\delta_{02}g \mid z)E(\delta_{02}g \mid z)\} + a_{11}a_{22}\{E(g^2 \mid z)E(\delta_{02} \mid z) - a_{12}^2 E(\delta_{02}g \mid z)E(g \mid z)\}] / \{a_{11}a_{22}E(g^2 \mid z)E(\delta_{02}^2 \mid z) - a_{12}^2 E^2(\delta_{02}g \mid z\} , \quad ...(2.3)$$

wetake $(\xi,\tau)' = (0,1)'$, consider the transformation

$$Z_1 = X_{1:N} \ , \ Z_2 = X_{n:N} - X_{1:N}$$
$$and \quad Z_i = \frac{X_{i-1:N} - X_{1:N}}{X_{n:N} - X_{1:N}}, \quad i = 3,4,\ldots,n.$$

Then

$$X_{1:N} = Z_1 , \ X_{n:N} = Z_1 + Z_2 \ , X_{i-1:N} = Z_1 + Z_2 Z_i ,$$

And the Jacobian of the transformation is $J = Z_2^{\,n-2}$.

Thus the joint pdf of $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ is given by

$$h(z_1,\ldots,z_n) = \{\prod_{i=1}^{n}(N - \sum_{j=1}^{i-1} r_j - i + 1)\} \ (1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2} \times \prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i} , \quad 0 < z_3 < \ldots < z_n < 1, 0 < z_1 + z_2 < 1.$$

Also, the joint pdf of $(Z_3,\ldots,Z_n)$ is given by

$$h_1(z_3,\ldots,z_n) = \{\prod_{i=1}^{n}(N - \sum_{j=1}^{i-1} r_j - i + 1)\} \int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2} \times \prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\} dz_1 dz_2, \quad 0 < z_3 < z_4 < \ldots < z_n < 1 .$$

Thus the conditional density of (Z_1, Z_2) given $(Z_3,\ldots,Z_n)$ is given by

$$h_2((z_1,z_2) \mid z_3,\ldots,z_n) = \{(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-2}\{(1-z_1-z_2 z_i)^{r_i}\}\} / \{ \int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-2}\{(1-z_1-z_2 z_i)^{r_i}\} dz_1 dz_2\} \quad ...(2.4)$$
$$, 0 < z_1 + z_2 < 1.$$

and

$$h_3(z_2 \mid z_3,\ldots,z_n) = \{ \int_0^{1-z_2} (1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\} dz_1\} / \{\int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\} dz_1 dz_2\} \quad ...(2.5)$$
$$, 0 < z_2 < 1.$$

Then

$$E(\delta_{01}\,\delta_{02}\,|\mathbf{z}) = E(z_1 z_2\,|\mathbf{z})$$
$$=\{\int_0^1\int_0^{1-z_2} z_1(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-1}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}/$$
$$\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}\;, \quad ...(2.6)$$

in view of (2.4). Also

$$E(\delta_{02}^{\;2}\,|\mathbf{z}) = E(z_2^{\;2}\,|\mathbf{z})$$
$$=\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}/$$
$$\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}\;, \quad ...(2.7)$$

In view of (2.5). Similarly,

$$E(\delta_{02}\,|\mathbf{z}) = E(z_2\,|\mathbf{z})$$
$$=\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-1}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}/$$
$$\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-2}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\} \quad ...(2.8)$$

In view of (2.5). Thus

$$w_1^{\;*} = \{\int_0^1\int_0^{1-z_2} z_1(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-1}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}/$$
$$\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\} \quad ...(2.9)$$

In view of (2.6) and (2.7) and

$$w_2^{\;*} = \{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\}/$$
$$\{\int_0^1\int_0^{1-z_2}(1-z_1)^{r_1}(1-z_1-z_2)^{r_n} z_2^{\,n-1}$$
$$\prod_{i=2}^{n-1}\{(1-z_1-z_2 z_i)^{r_i}\}\,dz_1 dz_2\} \quad ...(2.10)$$

In view of (2.7) and (2.8).

Therefore the MRE estimator $\boldsymbol{\delta}^* = (\delta_1^{\;*}, \delta_2^{\;*})'$ *of* $(\xi, \tau)'$ is given by

$$\delta_1^{\;*}(\mathbf{X}) = X_{1:N} - (X_{n:N} - X_{1:N})w_1^{\;*} \text{ and}$$
$$\delta_2^{\;*}(\mathbf{X}) = (X_{n:N} - X_{1:N})/w_2^{\;*},$$

where $w_1^{\;*}$ and $w_1^{\;*}$ are as given in (2.9) and (2.10) respectively.

If $N = 4, n = 3, r_1 = 1, r_2 = 0$, then the above estimator reduce to

$$\delta_1^*(\mathbf{X}) = X_{1:4} - (X_{3:4} - X_{1:4}) \left[\left\{\int_0^1 \int_0^{1-z_2} z_1(1-z_1) z_2^{\,2}\, dz_1 dz_2\right\} \Big/ \left\{\int_0^1 \int_0^{1-z_2} (1-z_1) z_2^{\,3}\, dz_1 dz_2\right\}\right]$$

$$= X_{1:4} - 4(X_{3:4} - X_{1:4})/15$$

$$= (19X_{1:4} - 4X_{3:4})/15$$

and

$$\delta_2^*(\mathbf{X}) = (X_{3:4} - X_{1:4}) \frac{\int_0^1 \int_0^{1-z_2} (1-z_1) z_2^{\,2}\, dz_1 dz_2}{\int_0^1 \int_0^{1-z_2} (1-z_1) z_2^{\,3}\, dz_1 dz_2}$$

$$= 4(X_{3:4} - X_{1:4})/15.$$

Remark 2.1. If $r_k = 0$, $k = 1,2,\ldots,n-1$ *and* $r_n = N - n$, then the above estimator reduces to

$$\delta_1^*(\mathbf{X}) = X_{1:n} - (X_{n:N} - X_{1:N})/n \text{ and}$$

$$\delta_2^*(\mathbf{X}) = (N+2)(X_{n:N} - X_{1:N})/n.$$

These estimators are same as the estimators obtained for type II right censored case(Leo Alexander, 2000)

Case (ii): Consider the location – scale invariant Linex loss function (Varian, 1975)

$$L(\xi,\tau;\delta) = \{e^{a(\delta_1-\xi)/\tau} - a(\delta_1 - \xi)/\tau - 1 + e^{b(\delta_2/\tau-1)} - b(\delta_2/\tau - 1)\},$$

in order to find $(w_1^{\,*}, w_2^{\,*})'$, take $\delta_{01}(\mathbf{X}) = X_{1:N}$ and $\delta_{02}(\mathbf{X}) = X_{n:N} - X_{1:N}$., consider

$$R(\boldsymbol{\delta} \mid \mathbf{z})$$

$$= E[\{e^{a\delta_1} - a\delta_1 - 1 + e^{b(\delta_2-1)} - b(\delta_2 - 1) - 1\} \mid \mathbf{z}]$$

$$= E[(e^{a(\delta_{01} - w_1\delta_{02})}) \mid \mathbf{z}] - aE(\delta_{01} \mid \mathbf{z}) + aw_1 E(\delta_{02} \mid \mathbf{z}) - + e^{-b} E(e^{b/w_2\, \delta_{02}} \mid \mathbf{z}) + b - b/w_2 E(\delta_{02} \mid \mathbf{z}) - 1$$

$$= \left\{\int_0^1 \int_0^{1-z_2} e^{a(z_1 - w_1 z_2)} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-2} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\} \Big/ \left\{\int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-2} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\}$$

$$- a\left\{\int_0^{1-z_2} z_1 (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-2} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_2\right\} \Big/ \left\{\int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-2} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\}$$

$$+ (aw_1 - b/w_2)\left\{\int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-1} \prod_{i=2}^{n-1} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\} \Big/ \left\{\int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-1} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\}$$

$$+ e^{-b}\left\{\int_0^1 \int_0^{1-z_2} e^{b/w_2 z_2} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-1} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\} \Big/ \left\{\int_0^1 \int_0^{1-z_2} (1-z_1)^{r_1} (1-z_1-z_2)^{r_n} z_2^{\,n-2} \prod_{i=2}^{n-1} \{(1-z_1-z_2 z_i)^{r_i}\}\, dz_1 dz_2\right\} + b - 2$$

Thus $(w_1^*, w_2^*)'$ is to be obtained as the value of $(w_1^*, w_2^*)'$ minimizing $R(\boldsymbol{\delta}\,|\,\mathbf{z})$.

Therefore the MRE estimator of $\boldsymbol{\delta}^* = (\delta_1^*, \delta_2^*)'$ *of* $(\xi, \tau)'$ is given by

$$\delta_1^*(\mathbf{X}) = X_{1:N} - (X_{n:N} - X_{1:N})\, w_1^*$$

and $\delta_2^*(\mathbf{X}) = (X_{n:N} - X_{1:N}) / w_2^*$.

Remark 2.2. If $r_k = 0$, $k = 1,2,\ldots,n-1$ *and* $r_n = N - n$, then the above estimator coincide with the estimators obtained for the type II right censored case of (Leo Alexander, 2000).

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