

The Fractional Schrödinger Equation and the Relativistic Schrödinger Equation: Twins



Physics

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ABSTRACT

As an approximate realization of the fractional quantum mechanics, we need to introduce the relativistic quantum mechanics. The fractional Schrödinger equation and the relativistic Schrödinger equation have many similar properties, such as their Hamiltonians are both Hermitian operators, the parity is conserved when the potential is even, and the angular momentum is conserved if the potential is central. They also have similar continuity equations of the probability. The twin equations should be studied at the same time. Meanwhile, we correct Laskin's continuity equation and this correction reveals a new way of the probability transportation—probability teleportation, which will be a new observable relativistic effect in quantum mechanics.

I. Introduction

In 2000, Laskin introduced the fractional quantum mechanics by revising the kinetic energy expression [1]. In [2], Laskin studied the property of the fractional Schrödinger equation, including the hermiticity of the Hamiltonian, the parity conservation law and the probability continuity equation etc. However, the probability continuity equation was wrong since a crucial term was missing. The new continuity equation with a source term indicates a new way for the probability transportation—teleportation.

What is more, Laskin did not point out the relation between the fractional Schrödinger equation and the real world. A natural question is, which particles have a fractional kinetic energy. If there are no Lakin's particles in our world, why do we need the fractional quantum mechanics?

We point out that the relativistic kinetic energy can be considered as an approximate realization of the fractional kinetic energy. The relativistic Schrödinger equation and the fractional Schrödinger equation have many similar properties, such as their Hamiltonians are both Hermitian operators, their parity is conserved when the potential is even, the angular momentum is conserved if the potential is central, and they also have similar continuity equations of the probability. The relativistic Schrödinger equation and the fractional Schrödinger equation should be studied at the same time as twins.

Next we will list the two equations, prove the hermiticity of their Hamiltonians and the conservation laws for the parity and the angular momentum, and present the correct continuity equation. We will study the crucial source term lost in [2] and the new phenomenon of the probability transportation—probability teleportation

II. The standard, fractional and relativistic Schrödinger equations

In this section, we will recall the standard and fractional Schrödinger equation, introduce the relativistic Schrödinger equation, and write them in a unified form.

1. The standard Schrödinger equation

In the standard quantum mechanics [3], the Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H\psi(\mathbf{r}, t), \quad (1)$$

where $\psi(\mathbf{r}, t)$ is a wave function defined in the 3 dimensional space \mathbb{R}^3 , changing dependent of time t . \mathbf{r} is a 3 dimensional vector. The Hamiltonian operator

$$H = T + V(\mathbf{r}) \quad (2)$$

is the summation of the operators of the kinetic energy and the potential energy of a particle.

The standard kinetic energy operator is

$$T = \frac{\mathbf{p}^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2, \quad (3)$$

where $\mathbf{p} = -i\hbar\nabla$ is the momentum operator. As usual, m is the mass of a particle and \hbar is the reduced Plank constant.

2. The fractional Schrödinger equation

The fractional Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H_\alpha \psi(\mathbf{r}, t), \quad (4)$$

where the fractional Hamiltonian operator

$$H_\alpha = T_\alpha + V(\mathbf{r}) \quad (5)$$

is the summation of the operators of the fractional kinetic energy and potential energy [2]. The fractional kinetic energy operator is expressed as

$$T_\alpha = D_\alpha |\mathbf{p}|^\alpha = D_\alpha (-\hbar^2 \nabla^2)^{\alpha/2}, \quad (6)$$

where the fractional parameter $1 < \alpha \leq 2$, and D_α is a coefficient dependent on α .

In the case $\alpha = 2$, $D_2 = 1/(2m)$, the fractional kinetic energy is the standard kinetic energy

$$T_2 = \frac{\mathbf{p}^2}{2m} = T. \quad (7)$$

Explicitly, the definition of the fractional kinetic energy operator is

$$T_\alpha \psi(\mathbf{r}) = \frac{1}{\sqrt{(2\pi\hbar)^3}} D_\alpha \int_{R^3} \bar{\psi}(\mathbf{p}) |\mathbf{p}|^\alpha \exp(i\frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3\mathbf{p}, \quad (8)$$

where

$$\bar{\psi}(\mathbf{p}) = \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \psi(\mathbf{r}) \exp(-i\frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3\mathbf{r} \quad (9)$$

is the wavefunction in the momentum space [3].

However, particles with a fractional kinetic energy do not exist in our world, which makes fractional quantum mechanics almost useless.

3. The relativistic Schrödinger equation

According to the special relativity [4], the kinetic energy should be revised as

$$T_r = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} . \tag{10}$$

For the case of low speed, the relativistic kinetic energy is approximately the summation of the rest energy and the classical kinetic energy ($\alpha = 2$)

$$T_r \approx mc^2 + \frac{\mathbf{p}^2}{2m} = mc^2 + T_2 , \tag{11}$$

and for the case of extremely high speed, where the rest energy can be neglected, the relativistic kinetic energy is the fractional kinetic energy with $\alpha = 1$

$$T_r \approx |\mathbf{p}| c = T_1 . \tag{12}$$

Generally speaking, if the speed of a particle increases from low to high, the relativistic kinetic energy T_r will approximately corresponds to a fractional kinetic energy T_α , whose parameter α changes from 2 to 1. Therefore the relativistic kinetic energy is an approximate realization of the fractional kinetic energy.

To relate the fractional Schrödinger equation to our world [5-7], we must introduce the relativistic Schrödinger equation [4]

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H_r \psi(\mathbf{r}, t) \tag{13}$$

where the relativistic Hamiltonian operator

$$H_r = T_r + V(\mathbf{r}) \tag{14}$$

is the summation of the operators of the relativistic kinetic energy and potential energy. The subscript r means the special relativity.

Explicitly, the definition of the relativistic kinetic energy operator is

$$T_r \psi(\mathbf{r}) = \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\psi}(\mathbf{p}) \exp(i \frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3 \mathbf{p} . \tag{15}$$

4. The unified form

The fractional Schrödinger equation and the relativistic Schrödinger equation can be written in a unified form

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H_{\beta} \psi(\mathbf{r}, t). \tag{16}$$

When the subscript $\beta = \alpha$, the above equation stands for the fractional Schrödinger equation (4), especially, when $\beta = 2$, the above equation is the standard Schrödinger equation (1), and when $\beta = r$, the above equation stands for the relativistic Schrödinger equation (13).

III. The hermiticity of the Hamiltonian operators

Theorem 1. The Hamiltonians H_{α} (5) and H_r (14) are both Hermitian.

Proof. We will prove the relativistic Hamiltonian H_r (14) only since the other one can be proved easily in the same way.

It is obviously that the potential operator is Hermitian,

$$V^+(\mathbf{r}) = V(\mathbf{r}). \tag{17}$$

In the Hilbert space, the inner product of two square integrable functions φ and χ is defined as

$$\langle \varphi, \chi \rangle = \int_{R^3} \varphi^*(\mathbf{r}) \chi(\mathbf{r}) d^3\mathbf{r}. \tag{18}$$

The Parseval's theorem on Fourier transform says that the inner product of two square integrable functions equals the inner product of their Fourier transforms, i.e.

$$\langle \varphi, \chi \rangle = \int_{R^3} \varphi^*(\mathbf{r}) \chi(\mathbf{r}) d^3\mathbf{r} = \int_{R^3} \bar{\varphi}^*(\mathbf{p}) \bar{\chi}(\mathbf{p}) d^3\mathbf{p}. \tag{19}$$

Now we say that the relativistic kinetic operator T_r is Hermitian, since

$$\langle T_r \varphi, \chi \rangle = \int_{R^3} \left(\sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\varphi}(\mathbf{p}) \right)^* \bar{\chi}(\mathbf{p}) d^3\mathbf{p} = \int_{R^3} \bar{\varphi}^*(\mathbf{p}) \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\chi}(\mathbf{p}) d^3\mathbf{p} = \langle \varphi, T_r \chi \rangle. \tag{20}$$

Therefore the relativistic Hamiltonian operator is Hermitian, since

$$(H_r)^+ = (T_r)^+ + (V(\mathbf{r}))^+ = T_r + V(\mathbf{r}) = H_r. \tag{21}$$

IV. The conservation Law in the fractional and relativistic quantum mechanics

Theorem 2. In the fractional or relativistic quantum mechanics, an observable is conserved if its operator and the Hamiltonian commute.

Proof.

Suppose that the observable has a Hermitian operator A. The average of this observable

$$\bar{A} = \langle \psi, A\psi \rangle = \int_{R^3} \psi^*(\mathbf{r}, t) A\psi(\mathbf{r}, t) d^3\mathbf{r} \tag{22}$$

varies in time t.

Its derivative is

$$\begin{aligned} \frac{d}{dt} \bar{A} &= \frac{d}{dt} \int_{R^3} \psi^*(\mathbf{r}, t) A\psi(\mathbf{r}, t) d^3\mathbf{r} \\ &= \int_{R^3} \frac{d}{dt} \psi^*(\mathbf{r}, t) A\psi(\mathbf{r}, t) d^3\mathbf{r} + \int_{R^3} \psi^*(\mathbf{r}, t) A \frac{d}{dt} \psi(\mathbf{r}, t) d^3\mathbf{r} \\ &= \int_{R^3} -\frac{1}{i\hbar} (H_\beta \psi)^*(\mathbf{r}, t) A\psi(\mathbf{r}, t) d^3\mathbf{r} + \int_{R^3} \psi^*(\mathbf{r}, t) A \frac{1}{i\hbar} H_\beta \psi(\mathbf{r}, t) d^3\mathbf{r} \\ &= \int_{R^3} -\frac{1}{i\hbar} \psi^*(\mathbf{r}, t) H_\beta A\psi(\mathbf{r}, t) d^3\mathbf{r} + \int_{R^3} \psi^*(\mathbf{r}, t) A \frac{1}{i\hbar} H_\beta \psi(\mathbf{r}, t) d^3\mathbf{r} \\ &= \frac{1}{i\hbar} \int_{R^3} \psi^*(\mathbf{r}, t) [A, H_\beta] \psi(\mathbf{r}, t) d^3\mathbf{r}. \end{aligned} \tag{23}$$

In the above deduction, the hermiticity of the Hamiltonian H_β is used.

Therefore an observable is conserved if its operator and the Hamiltonian commute,

$$[H_\beta, A] = 0. \tag{24}$$

Consequence 1. In the fractional or relativistic quantum mechanics, the parity is conserved if the potential is even, i.e., $V(-\mathbf{r}) = V(\mathbf{r})$.

Proof. Let's take the relativistic quantum mechanics as an example.

The parity operator P is defined as

$$P\chi(\mathbf{r}) = P\chi(-\mathbf{r}). \tag{25}$$

For any wavefunction $\psi(\mathbf{r})$, since

$$\begin{aligned}
T_r P \psi(\mathbf{r}) &= T_r \psi(-\mathbf{r}) = \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\psi}(-\mathbf{p}) \exp(i \frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3 \mathbf{p} \\
&= \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\psi}(\mathbf{p}) \exp(-i \frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3 \mathbf{p} \\
P T_r \psi(\mathbf{r}) &= P \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\psi}(\mathbf{p}) \exp(i \frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3 \mathbf{p} \\
&= \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \bar{\psi}(\mathbf{p}) \exp(-i \frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3 \mathbf{p},
\end{aligned} \tag{26}$$

we have

$$T_r P \psi(\mathbf{r}) = P T_r \psi(\mathbf{r}). \tag{27}$$

Therefore, the parity operator and the relativistic kinetic energy operator commute, i.e.

$$T_r P = P T_r \quad \text{or} \quad [T_r, P] = 0. \tag{28}$$

If the potential is even, that is

$$V(-\mathbf{r}) = V(\mathbf{r}), \tag{29}$$

we have

$$P V \psi(\mathbf{r}) = V(-\mathbf{r}) \psi(-\mathbf{r}) = V(\mathbf{r}) \psi(-\mathbf{r}) = V P \psi(\mathbf{r}), \tag{30}$$

$$P V = V P \quad \text{or} \quad [V, P] = 0. \tag{31}$$

Combining (28) and (31) we have

$$[H_r, P] = [T_r + V, P] = [T_r, P] + [V, P] = 0. \tag{32}$$

This completes the proof.

Consequence 2. In the fractional or relativistic quantum mechanics, the angular momentum is conserved if the potential is central, i.e., $V(\mathbf{r}) = V(r)$.

Proof. The angular momentum is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -\mathbf{p} \times \mathbf{r}. \tag{33}$$

In the coordinate representation, we have

$$\begin{aligned}
 \mathbf{L}V\psi(\mathbf{r}) &= \mathbf{r} \times \mathbf{p}V\psi(\mathbf{r}) = -i\hbar\mathbf{r} \times \nabla (V\psi(\mathbf{r})) \\
 &= -i\hbar\mathbf{r} \times (\nabla V(r))\psi(\mathbf{r}) - i\hbar\mathbf{r} \times V\nabla\psi(\mathbf{r}) \\
 &= 0 - i\hbar V\mathbf{r} \times \nabla\psi(\mathbf{r}) = V\mathbf{r} \times \mathbf{p}\psi(\mathbf{r}) \\
 &= V\mathbf{L}\psi(\mathbf{r}).
 \end{aligned}
 \tag{34}$$

Therefore we have

$$\mathbf{L}V = V\mathbf{L} \text{ or } [V, \mathbf{L}] = 0.
 \tag{35}$$

In the momentum representation [3], we have

$$\begin{aligned}
 \mathbf{L}T_\beta\bar{\psi}(\mathbf{p}) &= -\mathbf{p} \times \mathbf{r}T_\beta\bar{\psi}(\mathbf{p}) = -i\hbar\mathbf{p} \times \nabla_{\mathbf{p}} (T_\beta\bar{\psi}(\mathbf{p})) \\
 &= -i\hbar\mathbf{p} \times \nabla_{\mathbf{p}} (T_\beta(\mathbf{p}))\bar{\psi}(\mathbf{p}) - i\hbar\mathbf{p} \times T_\beta\nabla_{\mathbf{p}}\bar{\psi}(\mathbf{p}) \\
 &= 0 - i\hbar T_\beta\mathbf{p} \times \nabla_{\mathbf{p}}\bar{\psi}(\mathbf{p}) = -T_\beta\mathbf{p} \times \mathbf{r}\bar{\psi}(\mathbf{p}) \\
 &= T_\beta\mathbf{L}\bar{\psi}(\mathbf{p}).
 \end{aligned}
 \tag{36}$$

Here $\nabla_{\mathbf{p}}$ is the Laplacian operator in the momentum space. For example, we have

$$\nabla_{\mathbf{p}}\bar{\psi}(\mathbf{p}) = \mathbf{i} \frac{\partial}{\partial p_x} \bar{\psi}(\mathbf{p}) + \mathbf{j} \frac{\partial}{\partial p_y} \bar{\psi}(\mathbf{p}) + \mathbf{k} \frac{\partial}{\partial p_z} \bar{\psi}(\mathbf{p})$$

with $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$.

Therefore we have

$$\mathbf{L}T_\beta = T_\beta\mathbf{L} \text{ or } [T_\beta, \mathbf{L}] = 0.
 \tag{37}$$

Combining (35) and (37). We have

$$[\mathbf{H}, \mathbf{L}] = [\mathbf{T}, \mathbf{L}] + [\mathbf{V}, \mathbf{L}] = 0.
 \tag{38}$$

Therefore, the angular momentum is conserved for a central potential in either fractional or relativistic quantum mechanics. This completes the proof.

Furthermore, it is easy to see that

$$\begin{aligned}
 [\mathbf{H}, \mathbf{L}^2] &= [\mathbf{H}, \mathbf{L}] \cdot \mathbf{L} + \mathbf{L} \cdot [\mathbf{H}, \mathbf{L}] = 0 \\
 [\mathbf{H}, L_z] &= 0, [\mathbf{L}^2, L_z] = 0.
 \end{aligned}
 \tag{39}$$

Therefore, these three operators H, \mathbf{L}^2 and L_z are mutually commutable, and they form a complete set of observables, as in the standard quantum mechanics. This is important when we solve the fractional and relativistic Schrödinger equation with a central potential.

IV. The probability continuity equation

In this section, we will deduct the probability condition equation, find the condition for the source term to be zero, point out the different pictures of the probability transportation between the standard quantum mechanics and other quantum mechanics. The Laskin's mistake was corrected and new phenomenon based on the correct continuity equation is reported.

1. The general probability continuity equation

From the general Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H_\beta \psi(\mathbf{r}, t), \quad (40)$$

we have

$$\begin{aligned} i\hbar \psi^* \frac{\partial}{\partial t} \psi &= \psi^* T_\beta \psi + \psi^* V \psi, \\ i\hbar \psi \frac{\partial}{\partial t} \psi^* &= -\psi T_\beta \psi^* - \psi V \psi^*. \end{aligned} \quad (41)$$

Adding the above two equations, we have

$$i\hbar \frac{\partial}{\partial t} \psi^* \psi = \psi^* T_\beta \psi - \psi T_\beta \psi^*, \quad (42)$$

$$\frac{\partial}{\partial t} \psi^* \psi + \frac{i}{\hbar} (\psi^* T_\beta \psi - \psi T_\beta \psi^*) = 0. \quad (43)$$

By defining the fractional probability density and current density

$$\begin{aligned} \rho &= \psi^* \psi \\ \mathbf{j}_\beta &= \frac{i}{\hbar} (\psi^* \nabla^{-2} T_\beta \nabla \psi - \psi \nabla^{-2} T_\beta \nabla \psi^*), \end{aligned} \quad (44)$$

we have the probability continuity equation

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j}_\beta = I_\beta \quad (45)$$

with a source term

$$I_\beta = \frac{i}{\hbar} (\nabla \psi^* \nabla^{-2} T_\beta \nabla \psi - \nabla \psi \nabla^{-2} T_\beta \nabla \psi^*). \tag{46}$$

If $I(\mathbf{r},t) > 0$, there is a source at position \mathbf{r} and time t , which generates the probability; when $I(\mathbf{r},t) < 0$, there is a sink at position \mathbf{r} and time t , which destroys the probability.

Due to the hermiticity of the kinetic operator, from (42) we have

$$i\hbar \frac{\partial}{\partial t} \int_{R^3} \psi^* \psi d^3\mathbf{r} = \int_{R^3} \psi^* T_\beta \psi d^3\mathbf{r} - \int_{R^3} \psi T_\beta \psi^* d^3\mathbf{r} = 0, \tag{47}$$

which indicates that the total probability in the whole space is conserved.

Here is the picture of the probability transportation in the fractional quantum mechanics ($\alpha \neq 2$) or the relativistic quantum mechanics: the probability can disappear at some place and appear simultaneously at other places but the total probability does not change. In other words, some probability can be teleported from one place to another.

In fractional or relativistic quantum mechanics, the source term is zero only under a very special condition.

Theorem 3. For a free particle with a specific kinetic energy, the source term is zero.

Proof.

When $V(\mathbf{r}) = 0$, the general Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r},t) = T_\beta \psi(\mathbf{r},t). \tag{48}$$

For a specific kinetic energy, E , its solution is

$$\psi(\mathbf{r},t) = \int_\Omega C(\theta, \phi) \exp(i\mathbf{k} \cdot \mathbf{r}) \sin \theta d\theta d\phi \exp(-iEt / \hbar), \tag{49}$$

where Ω is the unit sphere, $(\mathbf{k}, \theta, \phi)$ is the spherical coordinate of the vector \mathbf{k} , and $C(\theta, \phi)$ is an arbitrary function.

Thus we have

$$T_\beta \psi(\mathbf{r},t) = E\psi(\mathbf{r},t) \tag{50}$$

$$T_\beta \psi^*(\mathbf{r},t) = E\psi^*(\mathbf{r},t). \tag{51}$$

In this case the source term vanishes

$$I = \frac{i}{\hbar} (\nabla \psi^* E \nabla \psi - \nabla \psi E \nabla \psi^*) = 0 \quad (52)$$

and the continuity equation has a sourceless form

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j}_\beta = 0, \quad (53)$$

with

$$\begin{aligned} \rho &= \psi^* \psi \\ \mathbf{j}_\beta &= -i \frac{\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*). \end{aligned} \quad (54)$$

For the simplest case

$$\psi(\mathbf{r}, t) = \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(-iEt / \hbar), \quad (55)$$

we have

$$\rho = 1, \mathbf{j}_\beta = \hbar \mathbf{k} / m, I_\beta = 0. \quad (56)$$

2. The continuity equation in the standard quantum mechanics

In the standard quantum mechanics ($\alpha = 2$), the kinetic energy is

$$T_\beta = T_2 = \frac{\mathbf{p}^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2 \quad (57)$$

From (46) we see the source term $I_\alpha(\mathbf{r}, t) = 0$ at any position and any time. The probability continuity equation has a sourceless form

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j}_2 = 0 \quad (58)$$

with

$$\begin{aligned} \rho &= \psi^* \psi \\ \mathbf{j}_2 &= -\frac{i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*). \end{aligned} \quad (59)$$

The sourceless continuity equation (58) indicates that the probability is conserved at every point in the space. They can never be generated nor destroyed. In other words, the probability cannot be

teleported. In scattering problem, what the detector measures is the current density at its location \mathbf{j}_2 .

3. The continuity equation in the fractional quantum mechanic

The continuity equation in the fractional quantum mechanics is

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j}_\alpha = I_\alpha, \tag{60}$$

with

$$\begin{aligned} \rho &= \psi^* \psi \\ \mathbf{j}_\alpha &= -iD_\alpha \hbar^{\alpha-1} (\psi^* (-\nabla^2)^{\alpha/2-1} \nabla \psi - \psi (-\nabla^2)^{\alpha/2-1} \nabla \psi^*) \end{aligned} \tag{61}$$

$$I_\alpha = -iD_\alpha \hbar^{\alpha-1} (\nabla \psi^* (-\nabla^2)^{\alpha/2-1} \nabla \psi - \nabla \psi (-\nabla^2)^{\alpha/2-1} \nabla \psi^*). \tag{62}$$

In [2] Laskin ever deducted the continuity equation, but he lost the source term I_α , or equivalently he supposed the source term is zero $I_\alpha = 0$. This was a crucial mistake.

For an example, take the wave function

$$\begin{aligned} \psi &= \psi_1 + \psi_2 \\ \psi_1(x, t) &= \exp(ik_1x) \exp(-iE_1t) \\ \psi_2(x, t) &= \exp(ik_2x) \exp(-iE_2t) \end{aligned} \tag{63}$$

with $k_1 > k_2 > 0$, $E_1 = D_\alpha (\hbar k_1)^\alpha$, $E_2 = D_\alpha (\hbar k_2)^\alpha$, which is a superposition of two solutions to the fractional Schrödinger equation for a free particle.

We have

$$\begin{aligned} I_\alpha &= -iD_\alpha \hbar^{\alpha-1} (\nabla \psi^* (-\nabla^2)^{\alpha/2-1} \nabla \psi - \nabla \psi (-\nabla^2)^{\alpha/2-1} \nabla \psi^*) \\ &= -iD_\alpha \hbar^{\alpha-1} \left((\psi_1^* + \psi_2^*)' (-\nabla^2)^{\alpha/2-1} (\psi_1 + \psi_2)' - (\psi_1 + \psi_2)' (-\nabla^2)^{\alpha/2-1} (\psi_1^* + \psi_2^*)' \right) \\ &= -iD_\alpha \hbar^{\alpha-1} (\psi_1^* (-\nabla^2)^{\alpha/2-1} \psi_2' + \psi_2^* (-\nabla^2)^{\alpha/2-1} \psi_1' - \psi_1' (-\nabla^2)^{\alpha/2-1} \psi_2^* - \psi_2' (-\nabla^2)^{\alpha/2-1} \psi_1^*) \\ &= -iD_\alpha \hbar^{\alpha-1} (k_1 k_2^{\alpha-1} \psi_1^* \psi_2 + k_1^{\alpha-1} k_2 \psi_2^* \psi_1 - k_1 k_2^{\alpha-1} \psi_1 \psi_2^* - k_1^{\alpha-1} k_2 \psi_2 \psi_1^*) \\ &= -iD_\alpha \hbar^{\alpha-1} (k_1 k_2^{\alpha-1} - k_1^{\alpha-1} k_2) (\psi_1^* \psi_2 - \psi_1 \psi_2^*) \\ &= 2D_\alpha \hbar^{\alpha-1} (k_1 k_2^{\alpha-1} - k_1^{\alpha-1} k_2) \sin((k_2 - k_1)x - (E_2 - E_1)t / \hbar), \end{aligned} \tag{64}$$

which is not zero unless $\alpha = 2$.

4. The continuity equation in relativistic Schrödinger equation

The continuity equation in the relativistic Schrödinger equation is

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j}_r = I_r, \quad (65)$$

with the probability density, current density, and the source term

$$\begin{aligned} \rho &= \psi^* \psi \\ \mathbf{j}_r &= ic \left(\psi^* \nabla^{-2} \sqrt{-\nabla^2 + m^2 c^2} / \hbar^2 \nabla \psi - \psi \nabla^{-2} \sqrt{-\nabla^2 + m^2 c^2} / \hbar^2 \nabla \psi^* \right) \\ I_r &= ic \left(\nabla \psi^* \nabla^{-2} \sqrt{-\nabla^2 + m^2 c^2} / \hbar^2 \nabla \psi - \nabla \psi \nabla^{-2} \sqrt{-\nabla^2 + m^2 c^2} / \hbar^2 \nabla \psi^* \right). \end{aligned} \quad (66)$$

The source term indicates that the probability can disappear from here and appear in somewhere else. In other words, the probability can teleported to the other place.

Since the relativistic kinetic energy (10) is true while the classical kinetic energy (7) is approximate, the probability continuity equation with the source term (65) is true while the popular probability continuity equation without the source term (58) is approximate. In other words, the probability teleportation is a new relativistic effect in quantum mechanics.

5. Probability teleportation in the scattering experiment

Let us have a look at the probability teleportation's effect on the scattering problem. We notice that in scatter problems the source term $I(\mathbf{r}, t) \neq 0$ at the detector's location though the potential at the detector may be zero. There are two facts that contradict the condition in Theorem 3.

1) The particle is not free. Since the potential $V(\mathbf{r})$ is not zero near the scattering target, the relation

$$T_\beta \psi(\mathbf{r}, t) = E \psi(\mathbf{r}, t) \quad (67)$$

does not hold.

2) The kinetic energy carried by the scattering particles may not be exactly the same, i.e. they may not be strictly monoenergetic in the experimental environment.

It is a new theoretical problem in quantum mechanics to develop a scattering model based on the probability continuity equation with the source term.

When the scattering model on the continuity equation with the source term is available, we need calculate the variation between the new model and current model, and design experiments to observe the new phenomenon of the probability teleportation.

Additionally, suppose a particle carries electric charge q . Multiplying continuity equation (65) with q , we have

$$\frac{\partial}{\partial t} q\rho + \nabla \cdot q\mathbf{j}_r = qI_r, \quad (68)$$

which reminds us that in quantum mechanics the electronic charge can disappear from here and appear somewhere else. We need to pay attention on the teleportation phenomenon of electrical charge as well.

VI. Conclusion

The relativistic Schrödinger equation can be introduced as an approximate realization of the fractional Schrödinger equation. They have many similar properties such as their Hamiltonians are both Hermitian operators, the parity is conserved when the potential is even, the angular momentum is conserved if the potential is central, and they have similar continuity equation of the probability. The twin equations should be studied at the same time [8,9]. Meanwhile we find the missing term in Laskin's continuity equation and this correction reveals a new way of the probability transportation-- probability teleportation, which is a new relativistic phenomenon in quantum mechanics.

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Appendix. The probability continuity equation in the momentum space.

In the real space, the general Schrodinger equation is

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H_\beta \psi(\mathbf{r}, t), \quad (69)$$

with the Hamiltonian operator

$$H_\beta = T_\beta + V = T_\beta(\mathbf{p}) + V(\mathbf{r}). \quad (70)$$

In the momentum space, the Schrodinger equation is

$$i\hbar \frac{\partial}{\partial t} \phi(\mathbf{p}, t) = H_\beta \phi(\mathbf{p}, t), \quad (71)$$

with the Hamiltonian operator [3]

$$H_{\beta} = T_{\beta}(\mathbf{p}) + V(i\hbar\nabla_{\mathbf{p}}). \quad (72)$$

Here

$$\phi(\mathbf{p}, t) = \frac{1}{\sqrt{(2\pi\hbar)^3}} \int_{R^3} \psi(\mathbf{r}, t) \exp(-i\frac{\mathbf{p}}{\hbar} \cdot \mathbf{r}) d^3\mathbf{r}$$

is the wavefunction in the momentum space at time t . $\nabla_{\mathbf{p}}$ is the gradient operator in the momentum space. The subscript \mathbf{p} means in the momentum space.

From (71), we have

$$i\hbar\phi^* \frac{\partial}{\partial t} \phi = \phi^* T_{\beta} \phi + \phi^* V \phi. \quad (73)$$

Taking complex conjugation, we have

$$-i\hbar\phi \frac{\partial}{\partial t} \phi^* = \phi T_{\beta} \phi^* + \phi(V\phi)^*. \quad (74)$$

Taking (73)-(74), we have

$$i\hbar \frac{\partial}{\partial t} (\phi\phi^*) = \phi^* V \phi - \phi(V\phi)^*, \quad (75)$$

$$\frac{\partial}{\partial t} (\phi\phi^*) + \frac{i}{\hbar} (\phi^* V \phi - \phi(V\phi)^*) = 1. \quad (76)$$

This can be expressed as a probability equation with sources

$$\frac{\partial}{\partial t} \rho_{\mathbf{p}} + \nabla_{\mathbf{p}} \cdot \mathbf{j}_{\mathbf{p}} = I_{\mathbf{p}}, \quad (77)$$

where the probability density, the current density, and the source term in the momentum space are defined as

$$\begin{aligned} \rho_{\mathbf{p}} &= \phi\phi^*, \\ \mathbf{j}_{\mathbf{p}} &= \frac{i}{\hbar} (\phi^* \nabla_{\mathbf{p}}^{-2} \nabla_{\mathbf{p}} (V\phi) - \phi \nabla_{\mathbf{p}}^{-2} \nabla_{\mathbf{p}} (V\phi)^*), \\ I_{\mathbf{p}} &= \frac{i}{\hbar} (\nabla_{\mathbf{p}} \phi^* \cdot \nabla_{\mathbf{p}}^{-2} \nabla_{\mathbf{p}} (V\phi) - \nabla_{\mathbf{p}} \phi \cdot \nabla_{\mathbf{p}}^{-2} \nabla_{\mathbf{p}} (V\phi)^*). \end{aligned} \quad (78)$$

If the potential energy is a harmonic oscillator potential [3]

$$V(\mathbf{r}) = \frac{1}{2} m\omega^2 \mathbf{r}^2 = -\frac{1}{2} m\omega^2 \hbar^2 \nabla_{\mathbf{p}}^2, \quad (79)$$

the probability continuity equation becomes

$$\frac{\partial}{\partial t} \rho_{\mathbf{p}} + \nabla_{\mathbf{p}} \mathbf{j}_{\mathbf{p}} = 0, \quad (80)$$

with

$$\begin{aligned} \mathbf{j}_{\mathbf{p}} &= \frac{i}{\hbar} (\varphi^* \nabla_{\mathbf{p}}^{-2} \nabla_{\mathbf{p}} (V\varphi) - \varphi \nabla_{\mathbf{p}}^{-2} \nabla_{\mathbf{p}} (V\varphi)^*) \\ &= -\frac{i}{2} m\omega^2 \hbar (\varphi^* \nabla_{\mathbf{p}} \varphi - \varphi \nabla_{\mathbf{p}} \varphi^*). \end{aligned} \quad (81)$$

Here ω is the angular frequency of the harmonic oscillator.

We see that the probability continuity equation in the momentum space is the same as in the real space. In other words, the probability teleportation also exists in the momentum space.

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