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Research Paper

Existence of Solutions for Generalized Vector Variational Inequalities

Mathematics

KEYWORDS : Generalized vector variational inequalities, FKKM theorem, hausdorff topological vector space, Convexity, Compactness.

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ABSTRACT

We introduce and study a class of generalized vector variational inequalities in the setting of hausdroff topological vector spaces. By using FKKM theorem, some new existence results of solutions for the generalized vector variational inequalities are obtained under some suitable conditions.

1. INTRODUCTION

Variational inequality theory plays an important role in many fields of science, engineering and economics. Because of their wide applicability, variational inequality problems have been generalized in various directions for the past several years. For details, we refer to [1–9] and the references therein.

The vector variational inequalities in a finite-dimensional Euclidean space were first introduced by Giannessi [10], which is the vector-valued version of the variational inequality of Hartman and Stampacchia [4]. Many authors studied several kinds of vector variational inequalities in abstract spaces [11, 12, 13].

Yu et al. [14] considered a more general form of weak vector variational inequalities and proved some new results on the existence of solutions of the new class of weak vector variational inequalities in the setting of hausdroff topological vector spaces. Very recently, Ahmad and Khan [15] introduced and considered weak vector variational-like inequalities with η -generally convex mapping and gave some existence results. On the other hand, Fang and Huang [16] studied some existence results of solutions for a class of strong vector variational inequalities in Banach spaces. In 2008, Lee et al. [17] introduced a new class of strong vector variational-type inequalities in Banach spaces. They obtained the existence theorems of solutions for the inequalities without monotonicity in Banach spaces by using Brouwer fixed point theorem and Browder fixed point theorem.

Motivated and inspired by the work mentioned above, in this paper we introduce and study a class of generalized vector variational inequalities in the setting of hausdroff topological vector spaces.

2. PRELIMINARIES

Let X and Y be two real hausdroff topological vector spaces, $K \subset X$ a nonempty, closed and convex subset, and $C \subset Y$ a closed, convex and pointed cone with apex at the origin. Recall that the hausdroff topological vector space Y is said to be an ordered hausdroff topological vector space denoted by (Y, C) if ordering relations are defined in Y as follows:

for all $x, y \in Y, x \leq y \Leftrightarrow y - x \in C,$

for all $x, y \in Y, x \nleq y \Leftrightarrow y - x \notin C$

If the interior $int\, C$ is nonempty, then the weak ordering relations in Y are defined as follows:

for all $x, y \in Y, x < y \Leftrightarrow y - x \in intC,$

for all $x, y \in Y, x \nless y \Leftrightarrow y - x \notin intC$

Let $L(X, Y)$ be the space of all continuous linear maps from X to Y and $T : X \to L(X, Y)$. We denote the value of $l \in L(X, Y)$ on $x \in X$ by (l, x) . Throughout this paper, we assume that $C(x): x \in K$ is a family of closed, convex and pointed cones of Y such that $intC(x) \neq \emptyset$ for all $x \in K$, and f is a mapping from $K \times K$ into Y.

In this paper, we consider the following vector variational inequality:

Generalized Vector Variational Inequality (for short GVVI): for each $z \in K$ and $\lambda \in (0,1]$, find $x \in K$ such that

$$\langle T(\lambda x + (1 - \lambda)z), y - x \rangle + f(y, x) \notin -intC(x), \forall y \in K$$

If $f(y, x) = 0$ and $C(x) = C$ for all $x, y \in K$ then GVVI reduces to the following model studied by Yu et al. [14].

Find $x \in K$ such that for each $z \in K$ and $\lambda \in (0,1]$,

$$\langle T(\lambda x + (1 - \lambda)z), y - x \rangle \notin -intC, \forall y \in K$$

For our main results, we need the following definitions and lemmas.

Definition 2.1 Let $T : K \to L(X,Y)$ be a mapping and $C = \bigcap_{x\in K} C(x) \neq \emptyset$. T is said to be monotone in C if and only if

$$\langle T(x) - T(y), x - y\rangle \in C, \forall x, y \in K$$

Definition 2.2 Let $T : K \to L(X,Y)$ be a mapping. We say that T is hemicontinuous if, for any given $x, y, z \in K$ and $\lambda \in (0,1]$, the mapping $t \mapsto \langle T\big(\lambda\big(x + (1-t)(y-x)\big) + (1-\lambda)z\big), y - x\rangle$ is continuous at 0^+.

Definition 2.3 A multivalued mapping $A : X \to 2^Y$ is said to be upper semicontinuous on X, if for all $x \in X$and for each open set G in Y with $A(x) \subset G$, there exists an open neighbourhood $O(x)$ of $x \in X$ such that $A(x') \subset G$ for all $x' \in O(x)$.

Lemma 2.1 [11] Let (Y, C) be an ordered topological vector space with a closed, pointed, convex cone C with $intC \neq \emptyset$. Then for every $x, y \in Y$, we have

(1) $y - x \in intC$ and $y \notin intC$ imply $x \notin intC$;
(2) $y - x \in C$ and $y \notin intC$ imply $x \notin intC$;
(3) $y - x \in -intC$ and $y \notin -intC$ imply $x \notin -intC$;
(4) $y - x \in -C$ and $y \notin -intC$ imply $x \notin -intC$.

Lemma 2.2 [18] Let M be a nonempty, closed, and convex subset of a hausdroff topological space, and $G: M \to 2^M$ a multivalued map. Suppose that for any finite set $\{x_1, x_2, \dots, x_n\} \subset M$, one has $co\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$ (i.e. F is a KKM mapping) and $G(x)$is closed for each $x \in M$ and compact for some $x \in M$, where co denotes the convex hull operator. Then $\bigcap_{x\in M} G(x) \neq \emptyset$.

Lemma 2.3 [19] Let X be a hausdroff topological space, $A_1, A_2, \dots\dots, A_n$ be nonempty compact convex subset of X. Then $co(\bigcup_{i=1}^n A_i)$ is compact.

Lemma 2.4 [20] Let X and Y be two topological spaces. If $A : X \to 2^Y$ is upper semicontinuous with closed values, then A is closed.

3. EXISTENCE RESULTS

Theorem 3.1 Let X be a hausdroff topological linear space, $K \subset X$ a nonempty, closed, and convex subset, and $\big(Y, C(x)\big)$ an ordered topological vector space with $intC(x) \neq \emptyset$ for all $x \in K$. Let $f : K \times K \to Y$ be affine mapping such that $f(x,x) = 0$ for each $x \in K$. Let $T : K \to L(X,Y)$ be hemicontinuous mapping. If $C = \bigcap_{x\in K} C(x) \neq \emptyset$ and T is monotone in C, then for each $z \in K$, $\lambda \in (0,1]$, the following statements are equivalent:

(i) find $x_0 \in K$, such that $\langle T_Z(x_0), y - x_0\rangle + f(y, x_0) \notin -intC(x_0)$, for all $y \in K$;
(ii) find $x_0 \in K$, such that $\langle T_Z(y), y - x_0\rangle + f(y, x_0) \notin -intC(x_0)$, for all $y \in K$

where T_Z is defined by $T_Z(x) = T(\lambda x + (1-\lambda)z)$ for all $x \in K$.

Proof Suppose that (i) holds. We can find $x_0 \in K$, such that

$$\langle T_Z(x_0), y - x_0\rangle + f(y, x_0) \notin -intC(x_0), \text{ for all } y \in K.$$

Since T is monotone, for each $x, y \in K$, we have

$$\langle T(\lambda y + (1-\lambda)z) - T(\lambda x + (1-\lambda)z), \lambda y + (1-\lambda)z - [\lambda x + (1-\lambda)z]\rangle \in C$$

Also, $\langle T_Z(y) - T_Z(x), y - x\rangle$
$= \frac{1}{\lambda}\langle T(\lambda y + (1-\lambda)z) - T(\lambda x + (1-\lambda)z), \lambda y + (1-\lambda)z - [\lambda x + (1-\lambda)z]\rangle \in C$

Hence T_Z is also monotone. That is

$$\langle T_Z(y) - T_Z(x_0), y - x_0\rangle \in C$$
$$\Rightarrow \langle T_Z(x_0) - T_Z(y), y - x_0\rangle \in -C$$

$$\Rightarrow \langle T_Z(x_0), y - x_0\rangle - \langle T_Z(y), y - x_0\rangle \in -C, \forall y \in K$$

Since $C = \bigcap_{x\in K} C(x)$, for all $y \in K$, we obtain

$$\langle T_Z(x_0), y - x_0\rangle + f(y, x_0) - \langle T_Z(y), y - x_0\rangle - f(y, x_0) \in -C \subset -C(x_0)$$

By Lemma 2.1,

$$\langle T_Z(y), y - x_0\rangle + f(y, x_0) \notin -intC(x_0), \forall y \in K$$

and so x_0 is a solution of (ii). Hence (i) implies (ii).

Conversely, suppose that (ii) holds. Then there exists $x_0 \in K$ such that

$$\langle T_Z(y), y - x_0\rangle + f(y, x_0) \notin -intC(x_0), \text{ for all } y \in K$$

For each $y \in K$, $t \in (0,1)$, we let $y_t = ty + (1-t)x_0$. Obviously, $y_t \in K$. It follows that

$$\langle T_Z(y_t), y_t - x_0\rangle + f(y_t, x_0) \notin -intC(x_0)$$

Since f is affine and $f(x_0, x_0) = 0$, we have

$$\langle T(\lambda(ty + (1-t)x_0) + (1-\lambda)z), ty + (1-t)x_0 - x_0\rangle + tf(y, x_0) \notin -intC(x_0)$$

$$\Rightarrow \langle T(\lambda(ty + (1-t)x_0) + (1-\lambda)z), t(y - x_0)\rangle + tf(y, x_0) \notin -intC(x_0)$$

That is

$$\langle T\big(\lambda\big(x_0 + t(y - x_0)\big) + (1-\lambda)z\big), y - x_0\rangle + f(y, x_0) \notin -intC(x_0)$$

Considering the hemicontinuity of T and letting $t \to 0^+$, we have

$$\langle T_Z(x_0), y - x_0\rangle + f(y, x_0) \notin -intC(x_0), \text{ for all } y \in K$$

Hence (ii) implies (i). This completes the proof.

Let K be a closed convex subset of a topological linear space X, and $\{C(x): x \in K\}$ a family of closed, convex, and pointed cones of a topological space Y such that $int\, C(x) \neq \emptyset$ for all $x \in K$. Throughout this paper, we define a set-valued mapping $\bar{C} : K \to 2^Y$ as follows:

$$\bar{C}(x) = Y \backslash \{-intC(x)\}, \forall x \in K$$

Theorem 3.2 Let X be a hausdroff topological linear space, $K \subset X$ a nonempty, closed, compact and convex subset, and $\big(Y, C(x)\big)$ an ordered topological vector space with $C(x) \neq \emptyset$, $\forall x \in K$. Let $f : K \times K \to Y$ be affine mapping such that $f(x,x) = 0$ for each $x \in K$. Let $T : K \to L(X,Y)$ be hemicontinuous mapping. Assume that the following conditions are satisfied

(i) $C = \bigcap_{x\in K} C(x) \neq \emptyset$ and T is monotone in C;

(ii) $\bar{C} : K \to 2^Y$ is an upper semicontinuous set-valued mapping.

Then for each $z \in K, \lambda \in (0,1]$, there exists $x_0 \in K$ such that

$$\langle T(\lambda x_0 + (1-\lambda)z, y - x_0)\rangle + f(y, x_0) \notin -intC(x_0), \text{ for all } y \in K$$

Proof For each $y \in K$, we denote $T_Z(x) = T(\lambda x + (1-\lambda)z)$, and define

$$F_1(y) = \{x \in K: \langle T_Z(x), y - x\rangle + f(y,x) \notin -intC(x)\},$$

$$F_2(y) = \{x \in K: \langle T_Z(y), y - x\rangle + f(y,x) \notin -intC(x)\}.$$

Then $F_1(y)$ and $F_2(y)$ are nonempty since $y \in F_1(y)$ and $y \in F_2(y)$. The proof is divided into the following three steps.

Step 1 First, we prove the following conclusion: F_1 is a KKM mapping. Indeed, assume that F_1 is not a KKM mapping; then there exist $u_1, u_2, \dots, u_m \in K$, $t_1 \geq 0, t_2 \geq 0, \dots, t_m \geq 0$ with $\sum_{i=1}^m t_i = 1$ and $w = \sum_{i=1}^m t_i u_i$ such that

$$w \notin \bigcup_{i=1}^m F_1(u_i), \qquad i = 1,2, \dots, m$$

That is,

$$\forall i = 1,2, \dots, m, \quad \langle T_Z(w), u_i - w\rangle + f(u_i, w) \in -intC(w)$$

Since f is affine and $f(w,w) = 0$, we have

$$\sum_{i=1}^m t_i \left[\langle T_Z(w), u_i - w\rangle + f(u_i, w)\right] \in -intC(w)$$

$$\Rightarrow \langle T_Z(w), \sum_{i=1}^{m} t_i u_i - w\rangle + f\left(\sum_{i=1}^{m} t_i u_i, w\right) \in -intC(w)$$
$$\Rightarrow \langle T_Z(w), w - w\rangle + f(w,w) \in -intC(w)$$
$$\Rightarrow 0 \in -intC(w)$$
Which is impossible and so $F_1 : K \to 2^K$ is a KKM mapping.

Step 2 Further, we prove that
$$\bigcap_{y\in K} F_1(y) = \bigcap_{y\in K} F_2(y)$$
In fact, if $x \in F_1(y)$, then $\langle T_Z(x), y - x\rangle + f(y,x) \notin intC(x)$. From the proof of Theorem 3.1, we know that T_Z is monotone in $C(x)$. It follows that
$$\langle T_Z(y) - T_Z(x), y - x\rangle \in C$$
$$\Rightarrow \langle T_Z(x) - T_Z(y), y - x\rangle \in -C$$
and so
$$\langle T_Z(x), y - x\rangle + f(y,x) - \langle T_Z(y), y - x\rangle - f(y,x) \in -C \subset -C(x)$$
By lemma 2.1, we have
$$\langle T_Z(y), y - x\rangle + f(y,x) \notin -intC(x)$$
and so $x \in F_2(y)$ for each $y \in K$. That is, $F_1(y) \subset F_2(y)$and so
$$\bigcap_{y\in K} F_1(y) \subset \bigcap_{y\in K} F_2(y)$$
Conversely, suppose that $x \in \bigcap_{y\in K} F_2(y)$. Then
$$\langle T_Z(y), y - x\rangle + f(y,x) \notin -intC(x), \forall y \in K$$
It follows from Theorem 3.1that
$$\langle T_Z(x), y - x\rangle + f(y,x) \notin -intC(x), \forall y \in K$$
That is, $x \in \bigcap_{y\in K} F_1(y)$ and so
$$\bigcap_{y\in K} F_2(y) \subset \bigcap_{y\in K} F_1(y)$$
which implies that
$$\bigcap_{y\in K} F_1(y) = \bigcap_{y\in K} F_2(y)$$
Step 3 Last, we prove that $\bigcap_{y\in K} F_2(y) \neq \emptyset$. Indeed, since F_1 is a KKM mapping, we know that, for any finite set $\{y_1, y_2, \dots, y_n\} \subset K$, one has
$$co\{y_1, y_2, \dots, y_n\} \subset \bigcup_{i=1}^{n} F_1(y_i) \subset \bigcup_{i=1}^{n} F_2(y_i)$$
This shows that F_2 is also a KKM mapping. Now, we prove that $F_2(y)$ is closed for all $y \in K$. Assume that there exists a net $\{x_\alpha\} \subset F_2(y)$ with $x_\alpha \to x \in K$. Then
$$\langle T_Z(y), y - x_\alpha\rangle + f(y, x_\alpha) \notin -intC(x_\alpha)$$
Using the definition of $\bar{C}$, we have
$$\langle T_Z(y), y - x_\alpha\rangle + f(y, x_\alpha) \in \bar{C}(x_\alpha)$$
Since f is continuous, it follows that
$$\langle T_Z(y), y - x_\alpha\rangle + f(y, x_\alpha) \to \langle T_Z(y), y - x\rangle + f(y,x)$$
Since $\bar{C}$ is upper semicontinuous mapping with close values, by lemma 2.4, we know that $\bar{C}$ is closed, and so
$$\langle T_Z(y), y - x\rangle + f(y,x) \in \bar{C}(x)$$
This implies that
$$\langle T_Z(y), y - x\rangle + f(y,x) \notin -intC(x),$$
and so $F_2(y)$ is closed. Considering the compactness of K and closeness of $F_2(y) \subset K$, we know that $F_2(y)$ is compact. By lemma 2.2, we have $\bigcap_{y\in K} F_2(y) \neq \emptyset$, and it follows that $\bigcap_{y\in K} F_1(y) \neq \emptyset$, that is, for each $z \in K$ and $\lambda \in (0,1]$, there exists $x_0 \in K$ such that
$$\langle T(\lambda x_0 + (1-\lambda)z, y - x_0)\rangle + f(y, x_0) \notin -intC(x_0), \text{ for all } y \in K$$
Thus GVVI is solvable. This completes the proof.

In the above Theorem, K is compact. In the following theorem, under some suitable conditions, we prove a new existence result of solutions for GVVI without the compactness of K.

Theorem 3.3 Let X be a hausdroff topological linear space, $K \subset X$ a nonempty, closed, and convex subset, and $(Y, C(x))$ be an ordered topological vector space with $C(x) \neq \emptyset$, $\forall x \in K$. Let $f : K \times K \to Y$ be affine mapping such that $f(x,x) = 0$ for each $x \in K$. Let $T : K \to L(X, Y)$ be hemicontinuous mapping. Assume that the following conditions are satisfied

(i) $C = \subset \bigcap_{x \in K} C(x) \neq \emptyset$ and T is monotone in C;

(ii) $\bar{C} : K \to 2^Y$ is an upper semicontinuous set-valued mapping;

(iii) there exists a nonempty compact and convex subset D of K and for each $z \in K, , \lambda \in (0,1]$, $x \in K \setminus D$, there exists $y_0 \in D$ such that
$$\langle T(\lambda y_0 + (1-\lambda)z, y_0 - x)\rangle + f(y_0, x) \in -intC(y_0)$$

Then for each $z \in K, \lambda \in (0,1]$, there exists $x_0 \in D$ such that
$$\langle T(\lambda x_0 + (1-\lambda)z, y - x_0)\rangle + f(y, x_0) \notin -intC(x_0), \text{ for all } y \in K$$

Proof By Theorem 3.1, we know that the solution set of the problem (ii) in Theorem 3.1 is equivalent to the solution set of following variational inequality: find $x \in K$,such that
$$\langle T(\lambda y + (1-\lambda)z), y - x\rangle + f(y, x) \notin -intC(x), \forall y \in K$$
For each $z \in K, \lambda \in (0,1]$; we denote $T_Z(x) = T(\lambda x + (1-\lambda)z)$. Let $G : K \to 2^D$ be defined as follows:
$$G(y) = \{x \in D: \langle T_Z(y), y - x\rangle + f(y, x) \notin -intC(x)\}, \forall y \in K$$
Obviously, for each $y \in K$,
$$G(y) = \{x \in K: \langle T_Z(y), y - x\rangle + f(y, x) \notin -intC(x)\} \cap D$$
Using the proof of the Theorem 3.2, we obtain that $G(y)$ is a closed subset of D. Considering the compactness of D and closedness of $G(y)$, we know that $G(y)$ is compact.

Now we prove that for any finite set $\{y_1, y_2, \dots, y_n\} \subset K$, one has $\bigcap_{i=1}^n G(y_i) \neq \emptyset$. Let $Y_n = \bigcup_{i=1}^n \{y_i\}$. Since Y is a real Hausdroff topological vector space, for each $y_i \in \{y_1, y_2, \dots, y_n\}$, $\{y_i\}$ is compact and convex. Let $N = co(D \cup Y_n)$. By lemma 2.3, we know that N is a compact and convex subset of K.

Let $F_1, F_2 : N \to 2^N$ be defined as follows:
$$F_1(y) = \{x \in N: \langle T_Z(x), y - x\rangle + f(y, x) \notin -intC(x)\}, \forall y \in N$$
$$F_2(y) = \{x \in N: \langle T_Z(y), y - x\rangle + f(y, x) \notin -intC(x)\} \forall y \in N$$
Using the proof of Theorem 3.2, we obtain
$$\bigcap_{y \in N} F_1(y) = \bigcap_{y \in N} F_2(y) \neq \emptyset$$
and so there exists $y_0 \in \bigcap_{y \in N} F_2(y)$.

Next we prove that $y_0 \in D$. In fact, if $y_0 \in K \setminus D$, then the assumption implies that there exists $u \in D$such that
$$\langle T(\lambda u + (1-\lambda)z), u - y_0\rangle + f(u, y_0) \in -intC(u)$$
which contradicts $y_0 \in F_2(u)$ and so $y_0 \in D$.

Since $\{y_1, y_2, \dots, y_n\} \subset N$ and $G(y_i) = F_2(y_i) \cap D$ for each $y_i \in \{y_1, y_2, \dots, y_n\}$, it follows that $y_0 \in \bigcap_{i=1}^n G(y_i)$. Thus, for any finite set $\{y_1, y_2, \dots, y_n\} \subset K$, we have $\bigcap_{i=1}^n G(y_i) \neq \emptyset$.

Considering the compactness of $G(y)$ for each $y \in K$, we know that there exists $x_0 \in D$ such that $x_0 \in \bigcap_{y \in K} G(y) \neq \emptyset$. Therefore, the solution set of GVVI is nonempty. This completes the proof.

4. CONCLUSION

In this work, we have considered generalized vector variational inequalities and prove some existence results in the setting of hausdorff topological vector spaces. The results presented in this article can be viewed as generalizations of some known vector variational inequality problems.

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