Research Paper

Volume : 4 | Issue : 9 | Sept 2015 • ISSN No 2277 - 8179

ON $\delta-\omega$ OPEN SETS

Mathematics

KEYWORDS : condense ideals, set, open set

Dr. S. Murugesan

Department of Mathematics Tejaa Shakthi Institute of Technology for Women, Coimbatore, Tamil Nadu 641659

ABSTRACT

We define and characterize sets, discuss the relation between $\delta-$ sets, $\delta-\omega$ sets and $\omega_\delta-$ sets and generalize some of the results.

1. Introduction and Preliminaries.

Throughout the paper, *cl*(A) and *int*(A) denote the closure and interior of A in a topological space (X, τ), respectively. An ideal I on X is a collection of subsets of X satisfying the following: (i)if $A \in I$ and $B \subset A$, then $B \in I$. (ii)if $A \in I$ and $B \in I$, then $A \cup B \in I$. A topological space (X, τ) together with an ideal I is called an ideal topological space and is denoted by (X,τ,I). for each subset A of X, A*(I, τ) = $\{ x \in X \mid U \cap A \notin I$ for every open set U containing $x \}$ is called the local function of A[7]with respect to I and τ. We simply write A* instead of A*(I, τ) in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [6] without mentioning it. Moreover, *cl**(A) = $A \cup A^*$ defines a Kuratowski closure operator [10] for a topology τ^* which is further then τ. An ideal I is a boundary ideal [6] or a condense ideal [4] if $\tau \cap I = \{\phi\}$. An ideal I is a completely condense ideal [4] if $PO(X) \cap I = \{\phi\}$, where PO(X) is the family of all preopen sets in (X, τ). A subset A is said to be preopen [8] if $A \subset \text{int}(cl(A))$. A subset A of a space (X,τ) is called $\omega-$ closed [5] if A contains all its condensation points. ($x \in X$ is a condensation point of A if $U \cap A$ is uncountable for every $U \in \tau(x)$). The complement of a $\omega-$closed set is a $\omega-$ open set. The family of all $\omega-$ opn sets denoted by τ_ω is topology on X, which is finer than τ. The interior and closure operator in (x,τ_ω) are denoted by int_ω and cl_ω respectively. If I_C is the ideal of all countable subsets of X, then for all subset A of X in the ideal space (X,τ,I_C), $A^*(I_C)$ is precisely the set of all condensation points of A [6]. Hereafter, we will denote $A^*(I_C)$ by simply $A_C{}^*$. The following lemmas will be useful in the sequel.

Lemma 1.1. *[6, Theorem 6.1]* if (X,τ,I) is an ideal space, then I is condense if and only if and $A \subset A^*$ for every open set A of X.

Lemma 1.2. *[96, Theorem 5] Let* (X,τ,I) be an ideal space and $A \subset X$. If $A \subset A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$.

2. $\delta-\omega-$sets.

A subset A of a topological space (X,τ) is a $\delta-$ set [3] if $\mathrm{int}(cl(A))\subset cl(\mathrm{int}(A))$. We will denote the family of all $\delta-$ sets in (X,τ) by $\delta(X,\tau)$ or simply $\delta(X)$. Clearly, every nowhere dense subset of X is $\delta-$ set. A subset A of a space (X,τ) is a $\delta-\omega-$ open set if $\mathrm{int}(c_\omega(A))\subset c_\omega(\mathrm{int}(A))$. The complement of a $\delta-\omega-$ open set is called a $\delta-\omega-$set. We will denote the family of all $\delta-\omega-$sets in (X,τ) by $\delta_\omega(X,\tau)$ or simply $\delta_\omega(X)$. A subset A of a space (X,τ) is a strong $-\beta-\omega$ open set (resp., *semi - ω- open set*) if $A\subset c_\omega(\mathrm{int}(c_\omega(A)))$ (resp., $A\subset c_\omega(\mathrm{int}(c_\omega(A)))$).the following Theorem 2.1 gives a decomposition of *semi - ω- open sets.*

Theorem 2.1. *Let* (X,τ) be a space. Then a subset of *A* of *X* is *semi - ω- open* if and only if it is both $\delta-\omega-$open and strong $-\beta-\omega$ open.

Proof. if *A* is *semi - ω- open, then* $A\subset c_\omega(\mathrm{int}(A)\subset c_\omega(\mathrm{int}(c_\omega(A))))$ and so *A* is $-\beta-\omega$ open. Moreover, $\mathrm{int}(c_\omega(A))\subset c_\omega(A)\subset c_\omega(c_\omega(\mathrm{int}(A)))\subset c_\omega(\mathrm{int}(A))$ and so *A* is $\delta-\omega-$open. Conversely, suppose *A* is both $\delta-\omega-$open and strong $\beta-\omega-$open. Then $A\subset c_\omega(\mathrm{int}(c_\omega(A))\subset c_\omega(c_\omega(\mathrm{int}(c_\omega(A))))=c_\omega(\mathrm{int}(A))$. Therefore, *A* is of *semi - ω- open.*

The following Theorem 2.2 gives a decomposition of $\alpha-\omega-$open sets.

Theorem 2.2. *Let* (X,τ) be a space. Then a subset of *A* of *X* is $\alpha-\omega-$open if and only if A is both $\delta-\omega-$open and *$pre-\omega-open.$.*

Proof. Suppose A is $\alpha-\omega-$open. Since every $\alpha-\omega-$ open set is *$semi-\omega-open.$*, by Theorem 2.1. A is $\delta-\omega-$open. Again , *A* is $\alpha-\omega-$open implies that $A\subset \mathrm{int}(C_\omega(\mathrm{int}(A)))\subset \mathrm{int}(C_\omega(A))$. Therefore, *A* is *$pre-\omega-open.$*. Conversely, suppose *A* is both $\delta-\omega-$open and strong *$pre-\omega-open.$*. Since *A* is *$pre-\omega-open.$*, we have $A\subset \mathrm{int}(c_\omega(A))$ and hence *A* is *$\alpha-\omega-open$.*

The following Example 2.5 shows that $\delta-\omega-$openness and $pre-\omega-$openness are independent concepts and also shows that $\delta-\omega-$openness and strong $\beta-\omega-$openness are independent concepts.

Volume : 4 | Issue : 9 | Sept 2015 • ISSN No 2277 - 8179

Example 2.3. *(a)Let* $X=\{a,b,c,d\}$ and $\tau=\{\phi,X,\{a,c\},\{d\},\{a,c,d\}\}$. Therefore, $Ic=\wp(X)$ and so $\tau_\omega=\wp(X)$. Then every subset of *X* is $\delta-\omega-$set and in particular, $A=\{b,c,d\}$ is a $\delta-\omega-$set, But $\text{int}(c_\omega(A))=\{d\}$ which implies that *A* is not $pre-\omega-open.$. Moreover, *A* is not strong $\beta-\omega-$open set.

(b) Consider ***X=R***, the set of all real numbers with the usual topology. Then $\tau_\omega=\tau^*(Ic)$ is the countable complement topology on **R**. if *A=I*, the set of all irrational numbers, then *A* is not $\delta-\omega-$open, since $\text{int}(c_\omega(A))=X$ and $c_\omega(\text{int}(A))=\phi$. Clearly, *A* is $pre-\omega-open.$ and also strong $\beta-\omega-$open set.

The following Theorem 2.4 shows that every $\delta-$set is a $\delta-\omega-$set.

Theorem 2.4. Let $\delta-\omega-$ be a open space where I_c be condense. then every $\delta-$set is a $\delta-\omega-$set and so $N\subset\delta(X)\subset\delta_\omega(X)$.

Proof. Suppose I_c is condense and *A* is a $\delta-$set. Then $\text{int}(cl(A))\subset cl(\text{int}(A))$. Now $\text{int}(C_\omega(A))\subset\text{int}(cl(A))\subset cl(\text{int}(A))=C_\omega(\text{int}(A))$, by Kemmas 1.1 and 1.2. Therefore *A* is $\delta-\omega-$set.

The following Example 2.5 shows that a $\delta-\omega-$set need not be a $\delta-$set.

Example 2.5. (a) *Let* $X=\{a,b,c,d\}$ and $\tau=\{\phi,X,\{a,c\},\{d\},\{a,c,d\}\}$. Therefore, $Ic=\wp(X)$ and so $\tau_\omega=\wp(X)$. Then every subset of *X* is $\delta-\omega-$set and in particular, $A=\{b,c,d\}$ is a $\delta-\omega-$set. Moreover, $cl(A)=X$ and so $\text{int}(A)=\{d\}$ and so $cl(\text{int}(A)=\{d\}$. Hence $\text{int}(cl(A))\not\subset cl(\text{int}(A))$ which implies that A is not a *A* is not a $\delta-$set.

The following Theorem 2.6 below shows that for completely condense ideals, $\delta-\omega-$sets Coincide with $\delta-$sets.

Theorem 2.6. *Let* (X,τ) be a space where *Ic* is completely condense ideals, $\delta-\omega-$sets Coincide with $\delta-$sets.

Proof. Suppose A is a $\delta-\omega-$set. Then $nt(C_{\omega}(A))\subset C_{\omega}(\text{int}(A))$. Since Ic is completely condense, by Lemmas 1.1 and 1.2. since Ic is completely condense, by Lemma 1.3, $Ic\subset N$ and so $A^{*}(N)\subset Ac^{*}$. Since $A^{*}(N)=cl(\text{int}(cl(A)))$ [10], $A^{*}(N)=cl(\text{int}(cl(A)))\subset Ac^{*}$. It follows that $\text{int}(cl(A))\subset\text{int}(Ac^{*})\subset\text{int}(C\omega(A))\subset cl(\text{int}(A))$. Therefore, A is a $\delta-$set. The reverse direction follows from Theorem 2.4, since every completely condense ideal is condense.

The following Theorem 2.7 gives a characterization of $\delta-\omega-$sets.

Theorem 2.7. *Let* (X,τ) be a space. Then $\text{int}(C\omega(A))\subset C\omega(\text{int}(A))$ and so $\text{int}(C\omega(A))\subset\text{int}(C\omega(\text{int}(A)))$. The converse is clear.

A subset A of an ideal space (X,τ) is said to be an $\omega_{\delta}-$sets in (X,τ) by $\omega_{\delta}(X,\tau)$ or simply $\omega_{\delta}(X)$. The following Theorem 2.8 shows that every $\delta-\omega-$set is a $\omega_{\delta}-$set. Theorem 2.9 below shows that If Ic is condense, the family of all $\delta-\omega-$sets coincide with the family of all $\omega_{\delta}-$sets.

Theorem 2.8. *Let* (X,τ) be a space. Then every $\delta-\omega-$set is a $\omega_{\delta}-$set.

Proof. Suppose $A\in\delta_{\omega}(X)$. Then $\text{int}(C_{\omega}(A))\subset C_{\omega}(\text{int}(A))$. Now $\text{int}(Ac^{*})\subset\text{int}(C_{\omega}(A))\subset C_{\omega}(\text{int}(A))=\text{int}(A)\cup(\text{int}(A))^{*}_{C}$.suppose $x\in\text{int}(A^{*}_{C})$. then $x\in\text{int}(A)\cup(\text{int}(A))^{*}_{C}$. Since $x\in\text{int}(A^{*}_{C})$. There is an open set G containing x such that $G\subset A^{*}_{C}$. If $x\notin(\text{int}(A))^{*}_{C}$, then there is an open set H containing x such that $H\cap\text{int}(A)$ is countable. Now $H\cap\text{int}(A)=H\cap(\text{int}(A)\cap A)=H\cap(\text{int}(A))\cap A$ is countable and so $((H\cap\text{int}(A))\cap A)^{*}_{C}=\phi$ which implies that $(H\cap\text{int}(A))\cap A^{*}_{C}=\phi$. Since $G\subset A^{*}_{C},(H\cap\text{int}(A))\cap G=\phi$ and so $(H\cap G)\cap\text{int}(A)=\phi$ which implies that $x\notin\text{int}(A)$,a contradiction to the fact that $x\in\text{int}(A)\cup(\text{int}(A))^{*}_{C}$. Therefore, $x\in(\text{int}(A))^{*}_{C}$ and so $\text{int}(A^{*}_{C})\subset(\text{int}(A))^{*}_{C}$. Here $A\in\omega_{\delta}(X)$.

Theorem 2.9. *Let* (X,τ) be a space. If Ic is condense, then $\delta_{\omega}(X)=\omega_{\delta}(X)$.

Proof. Suppose Ic is condense. By Theorem2.8, $\delta_{\omega}(X)=\omega_{\delta}(X)$. On other hand, if $A\in\omega_{\delta}(X)$, then $x\in\text{int}(A)\cup(\text{int}(A))^{*}_{C}$. Now $\text{int}(C_{\omega}(A))=\text{int}(A\cup A^{*}_{C})\subset\text{int}(A)\cup A^{*}_{C}$, since A^{*}_{C} is closed.

Since Ic is condense, $x \in \text{int}(A) \cup (\text{int}(A))^*_C$. Therefore, $\text{int}(C_\omega(A)) \subset (\text{int}(A))^*_C \cup A^*_C =$ $\text{int}(A) \cup A)^*_C$. $=A^*_C$ and so $\text{int}(C_\omega(A)) \subset (\text{int}(A))^*_C \subset C_\omega(\text{int}(A)$ which implies that $A \in \omega_\delta(X)$. Hence $\omega_\delta(X) \subset \delta_\omega(X)$

And so $\delta_\omega(X) = \omega_\delta(X)$.

Example 2.10. *[6, Example 2.4] Let* N be the set of all natural numbers with the topology τ generated by the base $\{\{2n\text{-}1, 2n\} | n \in N\}$. Then $\delta_\omega(X) = \omega_\delta(X) = \wp(X)$, but *Ic* is not condense.

A subset A of an ideal space (X,τ) is said to be *Ic*-dense if $A^*_C = X$ and *A* is said to be $\omega-$dense if $C_\omega(A) = X$. Clearly, every *Ic*-dense set is $\omega-$dense. The following sets Coincide with $\delta-\omega-$ sets.

Theorem 2.11. Let (X,τ) be a space and $A \subset X$ be a $\delta-\omega-$set. $A \subset B \subset A^*_C$ then B, A^*_C and B^*_C are $\delta-\omega-$sets.

Proof. $B \subset A^*_C$ implies that $B \subset C_\omega(A) \subset C_\omega(B)$ and so $C_\omega(A) = C_\omega(B)$. Since A is a $\delta-\omega-$set, it follows that $\text{int}(C_\omega(B)) = \text{int}(C_\omega(A)) \subset C_\omega(\text{int}(A)) \subset C_\omega(\text{int}(A))$ and so B is a $\delta-\omega-$set. Since $A \subset B \subset A^*_C$ implies that $A^*_C = B^*_C$, A^*_C and B^*_C are $\delta-\omega-$sets.

Corollary 2.12. Let (X,τ) be a space and $A \subset X$ be a $\delta-\omega-$set. Then the following hold.

(a) IF A is *- dense in itself, then A^*_C is a $\delta-\omega-$set.

(b) IF A is *Ic*- dense, then every superset of *A* is a $\delta-\omega-$set.

The following Theorem 2.13 gives a necessary condition for a set to be a $\delta-\omega-$set.

Theorem 2.13. Let (X,τ) be a space and $A \subset X$ be a $\delta-\omega-$set. Then $A = B \cup C$, where $B \in \tau$, $\text{int}(C\omega(C)) = \phi$ and $B \cap C = \phi$

Proof. Suppose *A* is a $\delta-\omega-$set. If $B = \text{int}(A)$ and $C = A - \text{int}(A)$, then B is an open set such that $A = B \cup C$ and $B \cap C = \phi$, Now $C \subset A$ implies $C_\omega(C) \subset C_\omega(A)$ and so

$\text{int}(C_\omega(C)) \subset \text{int}(C_\omega(C)) \subset C_\omega(\text{int}(A))$. Again $B \cap C = \phi$ implies that $B \cap C_\omega(C) = \phi$ which in turn implies that $B \cap \text{int}(C_\omega(C)) = \phi$ which implies that $C_\omega(B) \cap \text{int}(C_\omega(C)) = \phi$. Then $C_\omega(\text{int}(A)) \cap \text{int}(C_\omega(C)) = \phi$ and so $\text{int}(C_\omega(C)) = \phi$. This completes the proof.

REFERENCE

[1] A. A¸cikgoz, T. Noiri and S¸. Yuksel, On δ - I- open sets and decomposition of δ- continuity, Acta Math. Hungar., 102(4)(2004), 349 - 357. | [2] C. Chattopadhyay and C. Bandyopadhyay, On structure of δ- sets, Bull. Cal. Math. Soc, 83(1991), 281 - 290. | [3] J. Dontchev, M. Ganster and D. Rose, Ideal resolvability, Topology and its Applications, 93(1999), 1 - 16. | [4] B. Garai and C. Bandyopadhyay, Iδ- sets, I- o.h.i. spaces and I- resolvability, Kyungpook Math. J., 44(2004), 405 - 413. | [5] H. Hdeib, δ- closed mappings, Revista Colomb. De Matem., 16(1982), 65 -78. | [6] D. Jankovi´c and T.R. Hamlett, New Topologies from old via Ideals, Amer. Math. Monthly, 97(4)(1990), 295 - 310. | [7] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966. | [8] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53(1982), 47 -53. | [9] V. Renuka Devi, D. Sivaraj and T. Tamizh Chelvam, Codense and Completely codense ideals, Acta Math. Hungar., 108(3)(2005), 197 - 205. | [10] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, New York, 1960.