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On Semi- Slant Submanifolds of a Nearly Sasakian Manifold

Mathematics

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ABSTRACT

The object of the present chapter is to study the semi- slant submanifolds of a nearly Sasakian manifold. The conditions for some distributions on the semi- slant submanifolds of a nearly Sasakian manifold to be autoparallel has been explored. Further, the integrability conditions for some distributions has also been discussed.

. Introduction

Slant immersions in complex geometry were dened by Chen as a natural generalization of both holomorphic immersions and totally real immersions [1]. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [2]. Papaghiuc has introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifolds [3], such that the class of proper CR- subman-ifolds and the class of slant submanifolds appear as particular cases in the class of semi-slant submanifolds. The concept of the di erential geometry of semi- invariant or contact CR submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was given by Bejancu [9] and extended by many other ge-ometers([8],[7]). In 1999, Cabrerizo et.al.[4] have worked on the \`Semi- Slant Submanifolds of a Sasakian Manifold ". From this, we have motivated and study semi slant submanifolds of a nearly Sasakian manifold.

. Preliminaries

A $(2n+1)-$ dimensional Riemannian manifold (M,g) is said to be an almost contact manifold if there exist on M a (1,1) tensor field ϕ, a unique global non vanishing vector field ξ (called the structural vector field), a 1-form η and a compatible Riemannian metric g s.t.

$$(2.1)\quad \phi^2 = -I + \eta \otimes \xi,$$

$$\eta(\xi) = 1,\ \ \phi\xi = 0 \quad \eta o \phi = 0$$

$$(2.2)\quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

$$(2.3)\quad g(\phi X, Y) = -g(X, \phi Y),\ \ g(X,\xi) = \eta(X),$$

for all vector fields X, Y on M.

An almost contact metric manifold M with almost contact metric structure (ϕ,ξ,η,g) is called a nearly Sasakian manifold if

$$(2.4)\quad (\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y$$

Let M be a submanifold of a Riemannian manifold M with a Riemannian metric g then the Gauss and Weingarten formulae are given by

$$(2.5)\quad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$$

$$(2.6)\quad \tilde{\nabla}_X V = -A_V X + \nabla^{\perp} V,$$

for any $X, Y \in TM, V \in V^{\perp}M$; where $\tilde{\nabla}, \nabla$ and $\nabla^{\perp}$ are respectively the Riemannian, induced Levi- Civita and normal connections in M, M and the normal bundle $T^{\perp}M$ of M respectively then

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σ is the second fundamental form related to Weingarten map associated with V denoted by A_V, defined as

(2.7) $g(\sigma(X,Y),V) = g(A_V X,Y)$

For any $x \in M$, $X \in TM$ and $V \in T^{\perp}M$, we have

(2.8) $(\tilde{\nabla}_X \phi)Y = \{(\nabla_X H)Y - A_{LY}X - h\sigma(X,Y)\} + \{(\nabla_X L)Y + \sigma(X,HY) - l\sigma(X,Y)\}$

(2.9) $(\tilde{\nabla}_X \phi)V = \{(\nabla_X h)V - A_{lV}X - HA_V X)\} + \{(\nabla_X l)V + \sigma(X,hV) - LA_V X)\}$,

where ϕ is a (1,1) tensor field on M

and

(2.10) $\phi X = HX + LX$;

$HX \in T_x M$, $LX \in T_x^{\perp} M$

(2.11) $\phi V = hV + lV$;

$hV \in T_x M$, $lV \in T_x^{\perp} M$

(2.12) $(\nabla_X H)Y = \nabla_X HY - H\nabla_X Y$; $(\nabla_X L)Y = \nabla_X^{\perp} LY - L\nabla_X Y$;

(2.13) $(\nabla_X h)V = \nabla_X hV - h\nabla_X^{\perp} V$;

$(\nabla_X l)V = \nabla_X^{\perp} lV - l\nabla_X^{\perp} V$;

for any $X,Y \in TM$ and $V \in T^{\perp}M$.

From equation (2.3) and (2.10), we have

(2.14) $g(HX,Y) = -g(X,HY)$

for each $x \in M$, and $X,Y \in T_x M$.

3. Semi- Slant Submanifolds of Nearly Sasakian Manifolds

A submanifold M of an almost contact metric manifold M is said to be slant submanifold if for any $x \in M$ and $X \in T_x M$, linearly independent on ξ, the angle between ϕX and $T_x M$ is constant. The constant angle $\theta \in [0,\pi/2]$ is then called the slant angle or Wirtinger angle of M in M. If $\theta = 0$, the submanifold is invariant submanifold and if $\theta = \pi/2$, then submanifold is anti- invariant submanifold. If $\theta \neq 0,\pi/2$ then the submanifold is called a proper slant submanifold.

Definition 3.1 : A submanifold M is said to be a semi- slant submanifold of M if there exist two orthogonal distributions D_1 and D_2 on M such that at each point $x \in M$, the following conditions

(i) $T_x M$ admits an orthogonal direct decomposition i.e. $T_x M = D^1 \oplus D^2 \oplus \{\xi\}$

(ii) The distribution D_1 is an invariant distribution, i.e. $\phi(D_1) = D_1$

(iii) The distribution D_2 is slant with angle $\theta \neq 0$ hold;

where $\{\xi\}$ denotes the distribution spanned by the structure vector field ξ.

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It is obvious that invariant distribution of a semi- slant submanifold is a slant distribution with zero angle. Then it is obvious that semi- slant submanifolds are particular case of bi-slant submanifolds. Similarly we say M is anti- slant submanifold of M; if D_1 is an anti-invariant distribution of M i.e. $\phi(D_1) \subseteq T^{\perp}M$ and D_2 is slant with slant angle $\theta \neq 0$. Moreover, if $\theta = \pi/2$, then semi- slant submanifold is a semi- invariant submanifold.

For a semi- slant submanifold M, denote P_i the projection on the distribution D_i ; $i = 1,2$

$$(3.1) \quad X = P_1X + P_2X + \eta(X)\xi,$$

for any $X \in TM$,

where P_iX denotes the component of $X \in D_i$; $i = 1,2$.

Applying ϕ on (3.1), we have

$$(3.2) \quad \phi X = \phi P_1X + HP_2X + LP_2X.$$

It is easy to see that, for any $X \in TM$

$$(3.3) \quad \phi P^1X = HP^1X$$

$$and \;\; LP_1X = 0,$$

$$(3.4) \quad HP_2X \in D_2$$

In particular, we obtain

$$(3.5) \quad HX = \phi P_1X + HP_2X$$

and

$$(3.6) \quad LX = LP_2X$$

Lemma 3.2: If $\tilde{M}$ be a $(2n+1)$-dimensional nearly Sasakian manifolds, Then

$$(3.7) \quad \tilde{\nabla}_X\xi = -HX - H(\tilde{\nabla}_\xi\phi)X$$

and

$$(3.8) \quad \sigma(X,\xi) = -LX - L(\tilde{\nabla}_\xi\phi)X$$

Proof: Putting $Y = \xi$ in equation (2.4), we get

$$(3.9) \quad (\tilde{\nabla}_\xi\phi)X - \phi(\tilde{\nabla}_X\xi) = \eta(X)\xi - X$$

Operating ϕ, equation (3.9) becomes

$$(3.10) \quad \tilde{\nabla}_X\xi = -\phi X - \phi(\tilde{\nabla}_\xi\phi)X$$

Using equations (2.5)& (2.10), in (3.10) we have the results.

Lemma 3.3: Let M be a semi- slant submanifold of a nearly Sasakian manifold M,

$$(3.11) \quad g([X,Y],\xi) = 2g(HX,Y) + g(Y,H(\tilde{\nabla}_\xi\phi)X) - g(X,H(\tilde{\nabla}_\xi\phi)Y),$$

for any $X,Y \in D_2 \bigoplus D_1$.

Proof: From equations (2.12) and (3.9), we conclude the re

Proposition 3.4: Let M be a semi- slant submanifold of a nearly Sasakian manifold M and the slant distribution D_2 is integrable. Then D_2 is anti-invariant if

$$g(Y,H(\tilde{\nabla}_\xi\phi)X) = g(X,H(\tilde{\nabla}_\xi\phi)Y),$$

holds for all $X,Y \in D_2$.

Proof: For any $[X,Y] \in D_2$, Lemma (3.3) gives

$$g([X,Y],\xi) = 2g(HX,Y)$$

Further, if D_2 is integrable then

$[X,Y] \in D_2$, means $g([X,Y],\xi) = 0$

which implies $H = 0$ that means slant angle of D_2 is $\frac{\pi}{2}$

Lemma 3.5: Let M be a semi- slant submanifold of a nearly Sasakian manifold M. Then, for any $X, Y \in TM$, we have

(3.12) $P^1(\nabla_X \phi P^1 Y) + P_1(\nabla_X H\, P_2 Y) + P_1(\nabla_Y \phi P_1 X) + P_1(\nabla_Y H\, P_2 X)$

$$= \phi P_1(\nabla_X Y) + P_1 A_{LP^2Y} X + \phi P_1(\nabla_Y X) + P_1 A_{LP^2Y} Y - \eta(Y) P_1 X - \eta(X) P_1 Y$$

(3.13) $P_2(\nabla_X \phi P_2 Y) + P_2(\nabla_X H\, P_2 Y) + P_2(\nabla_Y \phi P_1 X) + P_2(\nabla_Y H\, P_2 X)$

$$= H\, P_{\,2}(\nabla_X Y) + P_2 A_{LP^2Y} X + 2h\sigma(X,Y) + H\, P_2(\nabla_Y X) + P_2 A_{LP^2X} Y - \eta(Y)\, P_2 X - \eta(X) P_2 Y$$

(3.14) $\eta(\nabla_X \phi P_2 Y) + \eta(\nabla_X H\, P_2 Y) + \eta(\nabla_Y \varphi P_1 X) + \eta(\nabla_Y H\, P_2 X) = H\, P_2(\nabla_X Y) + P_2 A_{LP^2Y} X + 2h\sigma(X,Y) + H\, P_2(\nabla_Y X) + P_2 A_{LP^2X} Y - \eta(Y)\, P_2 X - \eta(X) P_2 Y$

(3.15) $\sigma(\phi P_1 X, Y) + \sigma(\phi P_1 Y, X) + \sigma(HP_2 X, Y) + \sigma(HP_2 Y, X) + \nabla^{\perp}_{X} LP_2 Y + \nabla^{\perp}_{Y} LP_2 X$

$$= LP_2(\nabla_X Y) + LP_2(\nabla_X X) + 2l\sigma(X,Y)$$

Proof. Using Weingarten formula and equations (2.11), (3.1) & (3.2), equation (2.4) reduces into

(3.16) $\nabla_X \phi P_1 Y + \sigma(\phi P_1 Y, X) + \nabla_X HP_2 Y + \sigma(HP_2 Y, X) - A_{LP^2Y} X + \nabla^{\perp}_{X} LP_2 Y - \phi P_1(\nabla_X Y) - HP_2(\nabla_X Y) - LP_2(\nabla_X Y) - 2h\sigma(X,Y) - 2l\sigma(X,Y) + \nabla_Y \phi P_1 X + \sigma(\phi P_1 X, Y) + \nabla_Y HP_2 X + \sigma(HP_2 X, Y) - A_{LP^2X} Y + \nabla^{\perp}_{Y} LP_2 X - \phi P_1(\nabla_Y X) - HP_2(\nabla_Y X) - LP_2(\nabla_Y X)$

$$= 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi - \eta(Y) P_1 X - \eta(Y) P_2 X - \eta(X) P_1 Y - \eta(X) P_2 Y$$

By identifying the components on $D_1, D_2, \{\xi\}$ and $T^{\perp}M$, we get the desired results.

4. Parallel Distributions

Definition 4.1:The distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all $X, Y \in D$.

Definition 4.2: If a distribution D on M is autoparallel, then it is integrable and according to Gauss and Weingarten formula D is totally geodesic in M.

Definition 4.3: If D is parallel then the orthogonal complementary distribution $D^{\perp}$

is also parallel which implies that D is parallel if and only if $D^\perp$ parallel, therefore M is locally the product of the leaves of D and $D^\perp$

Proposition 4.4: Let M be a semi- slant submanifold of a nearly Sasakian manifold M.Then the distribution $D_1 \oplus \{\xi\}$ is autoparallel if and only if

$$(4.1) \quad \sigma(HX,Y) + \sigma(HY,X) = 2l\sigma(X,Y)$$

Proof. From equation (3.15), we observe that if $X, Y \in D_1 \oplus \{\xi\}$ then

$$\sigma(\phi X,Y) + \sigma(\phi Y,X) = LP_2(\nabla_X Y) + LP_2(\nabla_Y X) + 2l\sigma(X,Y)$$

Since distribution $D_1 \oplus \{\xi\}$ is autoparallel with respect to ∇ on M so for any $X, Y \in D_1 \oplus \{\xi\}$, $\nabla_X Y \in D_1 \oplus \{\xi\}$. Therefore

$$\sigma(\phi X,Y) + \sigma(\varphi Y,X) = 2l\sigma(X,Y)$$

and hence the result follows.

Corollary 4.5: The above distribution $D_1 \oplus \{\xi\}$ is also integrable.

Proof : The proof follows due to the fact that any distribution which is auto parallel on M, is integrable.

Proposition 4.6: Let M be a semi- slant submanifold of a nearly Sasakian manifold M. Then the distribution $D_2 \oplus \{\xi\}$ is autoparallel if and only if

$$P_1\nabla_X HY + \nabla_Y HX) = P_1(A_{LY}X + A_{LX}Y),$$

for any $X, Y \in D_2 \oplus \{\xi\}$

Proof : Obeying equations (3.5) and (3.6), the results are obtained.

Lemma 4.7: Let M be a submanifolds of a nearly Sasakian manifolds. Then for any $X, Y \in M$, we have

$$(4.7.1) \quad (\tilde{\nabla}_X\phi)Y = (\nabla_X\phi P_1)Y + (\nabla_X H\, P_2)Y + (\nabla_X LP_2)Y + \sigma(X,\phi P_1 Y) + \sigma(X, HP_2Y) - h\sigma(X,Y) - l\sigma(X,Y) - A_{LP_2Y}X$$

And

$$(4.7.2) \quad (\tilde{\nabla}_X\phi)V = (\nabla_X h)V + (\nabla_X l)V + \sigma(X,hV) + \phi P_1 A_V X + HP_2A_VX + LP_2A_VX - A_{lV}X$$

Proof : The result is followed by using equation (2.10) and the fact that D_1 is invariant.

Now the equations (2.4), (2.10) along with above equation (4.3) governs the following:

$$(4.2) \quad (\nabla_X\phi P_1)Y + (\nabla_X HP_2)Y + (\nabla_X LP_2)Y + \sigma(X,\phi P_1Y) + \sigma(X, HP_2X) - h\sigma(X,Y) - l\sigma(X,Y) - A_{LP_2Y} + (\nabla_Y\phi P_1)X + (\nabla_Y HP_2)X + (\nabla_Y LP_2)X + \sigma(Y,\phi P_1X) + \sigma(Y, HP_2X) - h\sigma(X,Y) - l\sigma(X,Y) - A_{LP_2X}Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y$$

On simplification, the equation (4.2) turns into

(4.3) $(\nabla_X \phi P_1)Y + (\nabla_X HP_2)Y - A_{LP_2Y}X - A_{LP_2X}Y - 2h\sigma(X,Y) + (\nabla_Y HP_2)X + (\nabla_Y \phi P_1)X$

$= 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y$

and

(4.4) $(\nabla_X LP_2)Y + (\nabla_Y LP_2)X + \sigma(X,\phi P_1 Y) + \sigma(X, HP_2 Y)$

$+\sigma(Y,\phi P_1 X) + \sigma(Y, HP_2 X) - 2l\sigma(X,Y) = 0$

Further, let $X, Y \in D_1 \oplus \{\xi\}$. Then equation (4.4) reduces into following

(4.5) $LP_2[X,Y] = 2LP_2(\nabla_X Y) - \sigma(X,\phi P_1 Y) - \sigma(Y,\phi P_1 X) + 2l\sigma(X,Y)$

If $D_1 \oplus \{\xi\}$ is integrable, i.e. $[X,Y] \in D_1 \oplus \{\xi\}$. And hence equation (4.5) gives us

(4.6) $2LP_2(\nabla_X Y) + 2l\sigma(X,Y) = \sigma(X,\phi P_1 Y) + \sigma(Y,\phi P_1 X)$

Again, equation (4.3) can be expressed as,

(4.7) $HP_2[X,Y] = (\nabla_X \phi P_1)Y + (\nabla_Y \phi P_1)X + 2HP_2(\nabla_X Y) + 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X$

But integrability of $D_1 \oplus \{\xi\}$ implies that $[X,Y] \in D_1 \oplus \{\xi\}$ for all $X, Y \in D_1 \oplus \{\xi\}$ and hence it is obvious that $HP_2[X,Y] = 0$

so equation (4.7) reduces into

(4.8) $(\nabla_X \phi P^1)Y + (\nabla_Y \phi P^1)X = \eta(X)Y + \eta(Y)X - 2g(X,Y)\xi - 2HP^2(\nabla_X Y)$

In view of the above discussion we state the following theorem:

Theorem 4.8: The distribution $D_1 \oplus \{\xi\}$ on a semi- slant submanifold is integrable if and only if

(4.8.1) $2LP_2(\nabla_X Y) + 2l\sigma(X,Y) = \sigma(X,\phi P_1 Y) + \sigma(Y,\phi P_1 X)$

and

(4.8.2) $(\nabla_X \phi P_1)Y + (\nabla_Y \phi P_1)X = \eta(X)Y + \eta(Y)X - 2g(X,Y)\xi - 2HP_2(\nabla_X Y)$

holds for all $X, Y \in D_1 \oplus \{\xi\}$

Corollary 4.9: The distribution $D_1 \oplus \{\xi\}$ is autoparallel if and only if

$$\sigma(Y,\phi P_1 X) + \sigma(X,\phi P_1 Y) = 2l\sigma(X,Y)$$

and

(4.10) $(\nabla_X \phi P_1)Y + (\nabla_Y \phi P_1)X = \eta(X)Y + \eta(Y)X - 2g(X,Y)\xi + \eta(X)\phi P_1 Y,$

holds for all $X, Y \in D_1 \oplus \{\xi\}$.

Proof : As $D_1 \oplus \{\xi\}$ is autoparallel and by using (4.6) & (4.8), the proof is straightforward and hence omitted.

REFERENCE

[1] B. Y. Chen: Geometry of slant submanifolds (Katholieke Universiteit Leuven, 1990). | [2] A. Lotta: Slant submanifolds in contact geometry, Bull. Math. Soc. Roumanie 39 (1996), 183- 198. | [3] N.Papaghiuc: Semi-slant submanifolds of a Kaehlerian manifold. An. Stiint. Al. I. Cuza. Univ. Iasi. 40 (1994), 55- 61. | [4] J. L. Cabrerizo, A. Carriazo: L. M. Fernndez and M. Fernndez,Semi-slant submani-folds of a sasakian manifold, Geometriae Dedicata 78(2): 183- 199, 1999. | | | | [5] C. Gherghe: Harmonicity on nearly trans- Sasakian manifolds, Demonstratio Mathe-matica Vol. XXXIII, 2000-No. 1,151-157. | [6] T. Koufogiorgos, C. Tsichlias: On the existence of a new class of contact metric manifolds,Canad. Math. Bull. Vol. 43 (4),(2000), 440- 447. | [7] R. Falleh Al- solamy: CR- Submanifolds of a nearly trans- Sasakian manifold, IJMMS 31(3): 2002, 167-175. | [8] Jeong- Sik Kim, Ximin liu and M. M. Tripathi : On semi- invariant submanifolds of nearly trans- Sasakian manifolds, Int. J. Pure & Appl. Math. Sci. Vol. 1(2004) pp. 15-34. | [9] A. Bejancu: Geometry of CR- Submanifolds, D. Reidel Publ. Co., Holland, 1986. |