Volume : 5 | Issue : 4 | April 2016 • ISSN No 2277 - 8179 | IF : 3.508 | IC Value : 69.48

Research Paper

Some Inference on Progressive Censored Gompertz Data Under Random Scheme

Medical Science

KEYWORDS : Random Scheme, Progressive Type-II Right Censoring, ML estimation, Bayes Estimation, and Bayes Prediction Bound Length.

Gyan Prakash Department of Community Medicine, S. N. Medical College, Agra, U. P., India

ABSTRACT

The censoring is useful in life testing experiments for time and cost restrictions. Especially, when some sample values at either or both the extremes might have been adulterated. But in some reliability experiment, the number of items dropped out the experiment cannot be pre-fixed and they are random. In such cases a random scheme is introduced inside the censoring criteria. The focused of the present article is discussed statistical inference based on a random scheme under progressive Type-II censored data. The Gompertz distribution is considered here as the underlying model for the study.

1. Introduction

In many life testing experiments, the experimenter may not be observing the lifetimes of all inspected units in life test. This may be because of time limitation and/or cost or material resources for data collection. Also, trimmed samples are widely used when some sample values at either or both extremes adulterated.

Prakash (2015 a) recently discussed about the Bayes prediction bound length under different censoring plans for Gompertz model. Based on progressive first-failure censoring plans Soliman et al. (2012) studied Bayes and frequentist estimators for two-parameter Gompertz distribution. Ismail (2010) discussed Bayes estimation for unknown parameters of Gompertz distribution and acceleration factors under partially accelerated life tests with Type-I censoring. They applied Bayesian approach to the estimation problem in the case of step stress partially accelerated life tests with two stress levels. Wu, et al. (2003) discussed point and interval estimations for Gompertz distribution under the Progressive Type-II censoring. Jaheen (2003) considered a Bayesian analysis in record statistics from Gompertz model.

Gompertz probability distribution has many useful applications in areas of technology, medical, biological, and natural sciences (especially in failure and survival analysis). This distribution also widely used in the model of human mortality and fit in actuarial tables. Gompertz (1825) introduced this model and its distribution and probability density function with scale parameter θ are defined as

$$F(x;\theta) = 1 - \exp\left(-\theta\, e^{x} + \theta\right); \theta > 0, x \ge 0 \qquad (1)$$

and

$$f(x;\theta) = \theta\, e^{x} \exp\left(-\theta\, e^{x} + \theta\right); \theta > 0, x \ge 0. \qquad (2)$$

There are several situations in life-testing, reliability experiments, and survival analysis in which units are lost or removed from the experiments while they are still alive. The loss may occur either out of control or pre-assigned. The out of control case can be happened when an individual under study drops out or so. The other case may occur because of limitation of funds or to save the time and cost.

But in some reliability experiment, the number of items dropped out the experiment cannot be pre-fixed and they are random. In such situations,

Volume : 5 | Issue : 4 | April 2016 • ISSN No 2277 - 8179 | IF : 3.508 | IC Value : 69.48

the progressive censoring schemes with random removals are suited best. In this paper, random removals have been used under Progressive Type-II censored Gompertz data. Some statistical inferences have presented and their performances are illustrated by an example based on real data.

2. Progressive Censoring with Random Scheme

The progressive censoring, appears to be a great importance in planned duration experiments in reliability studies. In many industrial experiments involving lifetimes of machines or units, experiments have to be terminated early and the number of failures must be limited for various reasons. Progressively Type-II censored sampling is an important method of obtaining data in such lifetime studies.

Let us suppose an experiment in which n independent and identical units $x_1, x_2, ..., x_n$ are placed on a life test at beginning time and first $m; (1 \le m \le n)$ failure items are observed. At the time of each failure occurring prior to termination point, one or more surviving units are removed from the test. Experiment is terminated at time of m^{th} failure, and all remaining surviving units are removed from the test.

Let $x_{(1)} \le x_{(2)} \le ... \le x_{(m)}$ are the lifetimes of completely observed units to fail and $R_1, R_2, ..., R_m; (m \le n)$ are the numbers of units withdrawn at these failure times (Prakash (2015 a)).

Following Prakash (2015 a), the joint probability density function under progressive Type-II censoring scheme is defined as

$$f_{\left(x_{(1:m:n)}, x_{(2:m:n)}, \ldots, x_{(m:m:n)}\right)} \left(\underline{x} \mid \theta\right) = C_p \prod_{i=1}^{m} f\left(x_{(i)}; \theta\right) \left(1 - F\left(x_{(i)}; \theta\right)\right)^{r_i};$$

$$\Rightarrow f_{\left(x_{(1:m:n)}, x_{(2:m:n)}, \ldots, x_{(m:m:n)}\right)} \left(\underline{x} \mid \theta\right) = C_p \, A_P^*\left(\underline{x}\right) \theta^m \exp\left(-\theta \, T_P^*\left(\underline{x}\right)\right); \tag{3}$$

where $A_p^*(\underline{x}) = \exp\left(\sum_{i=1}^{m} x_{(i)}\right)$ and $T_p^*(\underline{x}) = \sum_{i=1}^{m} \left(1 + r_i\right)\left(e^{x_{(i)}} - 1\right)$ and progressive normalizing constant $C_p = n\,(n - r_1 - 1)\,(n - r_1 - r_2 - 2) \ldots \left(n + 1 - \sum_{j=1}^{m-1} r_j - m\right)$.

Suppose an individual unit being removed from the test at $i^{th} \, (=1, 2, \ldots, m-1)$ failure, and is independent of the others with probability p i.e., r_i units removed at the i^{th} failure follows a Binomial distribution with parameters $n-m-\sum_{k=1}^{i-1} r_k$ and p. Hence,

$$P\left(R_i = r_i \mid R_{i-1} = r_{i-1},\ldots,R_1 = r_1\right) = \binom{n-m-\sum_{k=1}^{i-1} r_k}{r_i} p^{r_i}(1-p)^{n-m-\sum_{k=1}^{i} r_k} \quad \forall\, i = 1,2,\ldots,m-1. \tag{4}$$

Thus, $P(R=r)$ is now defined and obtained as

$$P(R=r) = P\left(R_1 = r_1\right) P\left(R_2 = r_2 \mid R_1 = r_1\right)\ldots P\left(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2},\ldots,R_1 = r_1\right)$$

$$\Rightarrow P(R=r) = \Omega\, p^{A}(1-p)^{B} \tag{5}$$

where $A = \sum_{i=1}^{m-1} r_i$, $B = (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i$ and $\Omega = \dfrac{(n-m)!}{(n-m-A)!\prod_{i=1}^{m-1} r_i!}$.

3. Point and Interval Estimation

The likelihood function is now defined as

$$L(\theta,p) = L\left(\theta;\underline{x} \mid R=r\right)\cdot P(R=r)$$

$$= C_p\, A_P^*\left(\underline{x}\right)\theta^{m}\exp\left(-\theta\, T_P^*(\underline{x})\right)\cdot \Omega\, p^{A}(1-p)^{B}$$

$$\Rightarrow L(\theta,p) = C^*\left(\theta^{m}\exp\left(-\theta\, T_P^*(\underline{x})\right)\right)\cdot\left(p^{A}(1-p)^{B}\right);\; C^* = C_p A_P^*\left(\underline{x}\right)\Omega. \tag{6}$$

The maximum likelihood (ML) estimators for the parameters θ and p are obtained as

$$\frac{\partial}{\partial\theta}\log L(\theta,p) = 0 \Rightarrow \hat{\theta}_{ML} = \frac{m}{T_P^*\left(\underline{x}\right)} \tag{7}$$

and

$$\frac{\partial}{\partial p}\log L(\theta,p) = 0 \Rightarrow \hat{p}_{ML} = \frac{A}{A+B}. \tag{8}$$

Now, the observed information matrix is defined and obtained as

$$I = \begin{pmatrix} \frac{\partial^2}{\partial\theta^2}\log L(\theta,p) & \frac{\partial^2}{\partial\theta\,\partial p}\log L(\theta,p) \\ \frac{\partial^2}{\partial p\,\partial\theta}\log L(\theta,p) & \frac{\partial^2}{\partial p^2}\log L(\theta,p) \end{pmatrix} = \begin{pmatrix} \frac{m}{\theta^2} & 0 \\ 0 & \frac{A}{p^2}+\frac{B}{(1-p)^2} \end{pmatrix}. \tag{9}$$

Hence, the variance covariance matrix is now approximated as

$$V = \begin{pmatrix} \frac{m}{\theta^2} & 0 \\ 0 & \frac{A}{p^2}+\frac{B}{(1-p)^2} \end{pmatrix}^{-1}$$

Thus the asymptotic distribution of ML estimator $\left(\hat\theta_{ML}, \hat p_{ML}\right)$ is given as

$$\begin{pmatrix} \hat\theta_{ML} \\ \hat p_{ML} \end{pmatrix} \sim N\left[\begin{pmatrix} \theta \\ p \end{pmatrix}, V\right]$$

Since, V involve unknown parameter θ and p. Hence, an estimate of $V\left(=\hat V\right)$ (say) is obtained by replacing parameters by its ML estimator

$$\hat V = \begin{pmatrix} \frac{m}{\hat\theta_{ML}^2} & 0 \\ 0 & \frac{A}{\hat p_{ML}^2}+\frac{B}{\left(1-\hat p_{ML}\right)^2} \end{pmatrix}^{-1}. \tag{10}$$

Thus the approximate $100(1-\varepsilon)\%$ confidence intervals for the parameters θ and p are derived as

$$\left(\hat\theta_{ML} \pm Z_{\varepsilon/2}\sqrt{\operatorname{Var}\left(\hat\theta\right)}\right) \quad \text{and} \quad \left(\hat p_{ML} \pm Z_{\varepsilon/2}\sqrt{\operatorname{Var}\left(\hat p\right)}\right)$$

where $\operatorname{Var}\left(\hat\theta\right)$ and $\operatorname{Var}\left(\hat p\right)$ are determined respectively from (10)

4. Bayes Estimation

A Bayesian procedure is applied in present section for estimating the parameters of the underlying model. Two-parameter Gamma distribution is selected here as conjugate prior for the unknown parameter θ, having probability density function

$$\pi_1(\theta) \propto \theta^{\beta-1} e^{-\alpha\theta}\,;\ \alpha > 0, \beta > 0, \theta \geq 0. \tag{11}$$

Similarly, prior density for unknown parameter p is considered here as the Beta distribution, having probability density function

$$\pi_2(p) \propto p^{\gamma-1}(1-p)^{\lambda-1}\,; 0 \leq p \leq 1. \tag{12}$$

The parameters θ and p are independent random variables, therefore, the joint prior density is obtained as

$$\pi(\theta, p) \propto \theta^{\beta-1} e^{-\alpha\theta} p^{\gamma-1}(1-p)^{\lambda-1}. \tag{13}$$

Based on Bayes theorem, the joint posterior and marginal posterior distributions are obtained respectively as

$$\pi^*(\theta, p) = J\,\theta^{m+\beta-1} e^{-\alpha^*\theta} p^{\gamma^*-1}(1-p)^{\lambda^*-1}, \tag{14}$$

$$\pi_1^*(\theta) = \frac{(\alpha^*)^{m+\beta}}{\Gamma(m+\beta)}\theta^{m+\beta-1} e^{-\alpha^*\theta} \tag{15}$$

and

$$\pi_2^*(p) = \frac{1}{B(\gamma^*, \lambda^*)} p^{\gamma^*-1}(1-p)^{\lambda^*-1} \tag{16}$$

where $\alpha^* = \alpha + T_P^*(\underline{x})$, $\gamma^* = \gamma + A$, $\lambda^* = \lambda + B$ and $J = \left\{\frac{\Gamma(m+\beta)}{(\alpha^*)^{m+\beta}} B(\gamma^*, \lambda^*)\right\}^{-1}$.

The resulting risk is often too sensitive to assumptions about behavior of tail of probability distribution when squared error loss function is taken as a measure of inaccuracy. Also, in some situations overestimation is more serious than underestimation, or vice-versa. To overcome this difficulty LINEX loss function (LLF) is a better choice and is defined as

$$L(\partial) = e^{a\partial} - a\partial - 1\,;\ \partial = \hat{\theta} - \theta.$$

Here 'a' is the shape parameter and $\hat{\theta}$ is any estimate of unknown parameter θ (See Prakash (2015 b)).

Following Prakash (2015 b), Bayes estimator corresponding to unknown parameter θ is denoted by $\hat{\theta}$, and obtained as

$$\hat{\theta}=-\frac{1}{a}\ln\ E\left\{e^{-a\theta}\right\}=\frac{m+\beta}{a}\ln\ \left\{1+\frac{a}{\alpha*}\right\}. \tag{17}$$

Similarly, the Bayes estimator corresponding to the parameter p say $\hat{p}$ is obtained by simplifying following equality

$$\hat{p}=-\frac{1}{a}\ln\ \int_{p}\frac{e^{-ap}}{B(\gamma*,\lambda*)}p^{\gamma*-1}(1-p)^{\lambda*-1}dp \tag{18}$$

The close form of $\hat{p}$ and the associated minimum posterior risk of Bayes estimators $\hat{\theta}$ and $\hat{p}$ do not exist. A numerical technique is applied herewith for drawing inferences.

5. Prediction Of The Future Records

Since $\underline{x}=\left(x_{(1)},x_{(2)},...,x_{(m)}\right)$ be first m observed failure items from a sample of size n under considered censoring plan for model (1). If we assume that $Y=\left(y_{(1)},y_{(2)},...,y_{(s)}\right)$ be another independent random sample of observations from same model. Then the Bayes predicative density of future observation Y is denoted by $h(Y\,|\,\underline{x})$ and obtained by simplifying the following relation

$$h\left(Y\,|\,\underline{x}\right)=\int_{\theta}f\ (y;\theta).\ \pi_1^*\left(\theta\,|\,\underline{x}\right)d\theta$$

$$=\frac{(\alpha*)^{m+\beta}}{\Gamma(m+\beta)}e^{y}\int_{\theta}\exp\ \left(-\theta\ e^{y}+\theta-\alpha*\theta\right)\theta^{m+\beta}\ d\theta$$

$$\Rightarrow h\left(Y\,|\,\underline{x}\right)==\frac{(\alpha*)^{m+\beta}(m+\beta)}{\left(e^{y}+\alpha*-1\right)^{m+\beta+1}}e^{y}. \tag{19}$$

Let the lower and upper Bayes prediction limits are denoted by l_1 and l_2 for random variable Y and $(1-\varepsilon)$ is called confidence prediction coefficient. Then one-sided Bayes prediction lower and upper limits are obtained by solving following equality

$$\Pr\ \left(Y\leq\ l_1\right)=\frac{\varepsilon}{2}=\Pr\ \left(Y\geq l_2\right). \tag{20}$$

Using equation (19) in (20) the one sided Bayes prediction lower and upper limits of Y are obtained as

$$l_1 = \ln\left\{1+\alpha^*\left(\varepsilon^*-1\right)\right\} ; \varepsilon^* = \left(1-\frac{\varepsilon}{2}\right)^{-1/(m+\beta)}$$

and

$$l_2 = \ln\left\{1+\alpha^*\left(\varepsilon^{**}-1\right)\right\} ; \varepsilon^{**} = \left(\frac{\varepsilon}{2}\right)^{-1/(m+\beta)} .$$

Hence, the one-sided Bayes prediction bound length is obtained as

$$L = l_2 - l_1 .$$

6. Numerical Analysis

The performance of the proposed procedures is studied by a numerical illustration based on a real data set considered by King et al. (1979) for tumor-free days of 30 rats fed with unsaturated diet. These data are also presented by Lee (1992) and studied by Chen (1997), Wu et al. (2003) and Prakash (2015 a) also. The data are given in the Table 1.

Table 1: Tumor - Free Days of 30 Rats

112	68	84	109	153	143	60	70	98	164
63	63	77	91	91	66	70	77	63	66
66	94	101	105	108	112	115	126	161	178

We carry out this analysis by considering the given data of size $m(=5,10,15)$ with selected progressive censoring scheme (presented in Table 2). The maximum likelihood estimates for both parameters are presented in Table 3. It is observed that the values of ML estimator are increasing as censored sample size increases.

Table 2: Different Progressive Censoring Scheme

Case	m	$R_i\ ; i=1,2,...,m$
1	5	1 2 1 0 1
2	10	1 0 0 3 0 0 1 0 0 1
3	15	1 0 2 0 0 1 0 2 0 0 0 1 0 0 1

Table 3: Maximum Likelihood Estimates

	$m\downarrow$		
$n=30$	5	10	15
$\hat{\theta}_{ML}$	0.4767	0.4799	0.4805
$\hat{p}_{ML}$	0.7698	0.7721	0.7793

The risk corresponding to Bayes estimator $\hat{\theta}$ under LLF are obtained and presented in the Table 4. The selected values of shape parameter 'a' are 0.50 and 1.00. While the values of prior parameters are selected as (β,α) $=(0.50,0.70),(1.00,1.00),(2.50,1.58),(5,2.30),(10,3.16)$ and $(0,0)$. Here, the criteria behind the selection of these prior parametric values is that the prior variance should be unity. Also, $\beta=\alpha=0$ reflect the study under non-informative (Jeffrey's) prior. Hence, all the results should be valid for both informative and non-informative priors.

Table 4: Risk Corresponding to Bayes Estimator $\hat{\theta}$

$n=30$		$\leftarrow(\beta,\alpha)\rightarrow$					
$a\downarrow$	$m\downarrow$	0.50, 0.70	1.00, 1.00	2.50, 1.58	5.00, 2.30	10, 3.16	0,0
0.50	5	0.6426	0.6361	0.6133	0.5918	0.5646	0.5793
	10	0.7878	0.7802	0.7513	0.7252	0.7027	0.7216
	15	1.1031	1.0928	1.0520	1.0148	0.9955	1.0345
1.00	5	0.6680	0.6576	0.6265	0.5990	0.5678	0.6065
	10	0.8196	0.8067	0.7684	0.7345	0.7073	0.7570
	15	1.1514	1.1328	1.0770	1.0288	1.0020	1.0888

It is noted that the risk increases as the censored sample size m increases. Similar trend also has seen when the values of shape parameter increases. The opposite trend has seen when the set of prior parametric value increases. However, the magnitude of risks is nominal.

Similarly, the risk corresponding to Bayes estimator $\hat{p}$ under LLF are obtained and presented in Table 5, for the similar set of selected parametric values. However, the selection of these prior parametric values does not provide unity variance for prior defined in equation 12.

Table 5: Risk Corresponding to Bayes Estimator $\hat{p}$

$n=30$		$\leftarrow(\gamma,\lambda)\rightarrow$					
$a\downarrow$	$m\downarrow$	0.50, 0.70	1.00, 1.00	2.50, 1.58	5.00, 2.30	10, 3.16	0,0
0.50	5	0.7626	0.7321	0.7133	0.6978	0.6546	0.6413
	10	0.8578	0.7692	0.7334	0.7129	0.6727	0.6606
	15	1.1911	1.1218	1.0519	1.0198	0.9884	0.9315
1.00	5	0.8923	0.8673	0.8452	0.8171	0.7965	0.8509
	10	1.1044	0.9907	0.9588	0.9348	0.8877	0.8735
	15	1.2947	1.2736	1.2517	1.2041	1.1974	1.0907

Similar properties have seen for risks of Bayes estimator $\hat{p}$ under LLF as discussed earlier for the Bayes estimator $\hat{\theta}$. It is further noted that the risks magnitude is larger for Bayes estimator $\hat{p}$ as compared to $\hat{\theta}$.

The Bayes prediction bound length and confidence interval are presented in Table 6, for $\varepsilon = 99\%, 95\%, 90\%$. It is noted that when confidence level ε decreases the length of intervals tends to be closer. The bounds length tends to be closer also as the set of prior parameters increases when other parametric values are considered fixed. A decreasing trend has been seen in length when censored sample size increases.

It is noted further that the bound length of based on ML procedure is closer as compared to Bayes procedure.

Table 6: Central Coverage Bayes Prediction Bound Lengths under Progressive Type-II Censoring Plans (Based on Simulation Data)

$n=30$		$\leftarrow(\beta,\alpha)\rightarrow$ Bayes Procedures						ML Procedure	
$m\downarrow$	ε	0.50, 0.70	1.00, 1.00	2.50, 1.58	5.00, 2.30	10, 3.16	0, 0	θ	p
5	99%	3.0490	2.9626	2.9009	2.8413	2.8137	3.0331	2.3491	2.9136
	95%	2.2147	2.1088	2.0871	2.0187	1.9837	2.1667	2.0147	2.8458
	90%	1.8071	1.7506	1.7072	1.6685	1.5936	1.7684	1.4571	1.7345
10	99%	2.9223	2.9141	2.8682	2.8046	2.7459	2.9996	1.9023	2.8904
	95%	2.1736	2.0752	2.0412	1.9896	1.9599	2.1432	1.8736	2.7132
	90%	1.7740	1.6914	1.6682	1.6383	1.5590	1.7489	1.7164	1.8004
15	99%	2.4851	2.4815	2.4734	2.4135	2.4039	2.9672	1.8851	2.1805
	95%	2.0193	2.0041	1.9908	1.9413	1.8873	2.1197	1.7193	2.1001
	90%	1.7413	1.6114	1.5433	1.5307	1.5191	1.7301	1.5433	1.3404

Volume : 5 | Issue : 4 | April 2016 • ISSN No 2277 - 8179 | IF : 3.508 | IC Value : 69.48

References

1. Chen. Z. (1997). Parameter estimation of the Gompertz population. Biometrical Journal, 39 (1), 117–124.
2. Gompertz, B. (1825). On the nature of the function expressive of the Law of Human Mortality, and on a new mode of determining the value of life contingencies. Philosopher Transaction Royal Society of London, 115, 513-585.
3. Ismail, A. A. (2010). Bayes estimation of Gompertz distribution parameters and acceleration factor under partially accelerated life tests with Type-I censoring. Journal of Statistical Computation and Simulation, 80 (11), 1253-1264.
4. Jaheen Z. F. (2003). A Bayesian analysis of record statistics from the Gompertz model. Applied Mathematics and Computation, 145 (2), 307-320.
5. King, M. M., Bailey, D. M., Gibson, D. D., Pitha, J. V. & McCay, P. B. (1979). Incidence and growth of mammary tumors induced by 7, 12-dimethylbenz anthracene as related to the dietary content of fat and antioxidant. Journal of the National Cancer Institute, 63 (3), 657–663.
6. Lee, E. T. (1992). Statistical methods for survival data analysis-2nd edition. Wiley, New York.
7. Prakash, G. (2015 a). A comparative study based on Bayes estimation under different censoring criterion. Journal of Data Science, 13 (2), 261 – 280.
8. Prakash, G. (2015 b). A comparative study under progressively first failure censored Rayleigh data. Journal of Reliability and Statistical Studies, 8 (1), 147 - 157.
9. Soliman, A. A., Abd-Ellah, A. H., Abou-Elheggag, N. A. & Abd-Elmougod, G. A. (2012). Estimation of the parameters of life for Gompertz distribution using progressive first-failure censored data. Computational Statistical Data Analysis, 56 (8), 2471-2485.
10. Wu, S. J., Chang, C. T. & Tsai, T. R. (2003). Point and interval estimations for the Gompertz distribution under progressive Type-II censoring. Metron, 61, 403–418.