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Solving Delay Differential Equations With Constant Lags Using Rkham Method

Mathematics

KEYWORDS : Delay differential equations, constant delay, multiple delays, IVP, RKAM, RKHaM.

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ABSTRACT

This paper presents RK method based on Harmonic mean for solving Delay differential equations with constant delays. The delay term is approximated by using linear and Lagrange interpolation. The effectiveness of this method has been illustrated with examples. The numerical results are also compared with RK method based on Arithmetic mean.

INTRODUCTION

Delay differential equations (DDEs) arise in population dynamics [1], control systems [2], chemical kinetics [3] and in many areas of Science and engineering. Recently there has been great interest in finding the numerical solutions of DDEs. Many numerical methods have been suggested for solving DDEs of first order as in [4-6]. Paul and Baker [7] discussed about the determination of stability region of RK method for DDEs. Hu et al. [8] considered the stability of RK methods for DDEs with multiple delays.

Evans and Sanugi [9] have introduced the concept of fourth order RK formulae based on HaM. Murugesan et al. [10] have applied RKCeM to solve second order IVPs.

In this paper the fourth order RKHaM method has been discussed for obtaining the numerical solution of first order DDEs. Here we consider the first order DDEs in the following form:

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t > t_0$$

$$y(t) = \Phi(t), \quad t \le t_0$$

if it has one delay only, or

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_n)), \quad t > t_0$$
$$y(t) = \Phi(t), \quad t \le t_0$$

if it has more than one delay term where $\Phi(t)$ is the initial function. Here the delay terms $\tau, \tau_1, \tau_2, \ldots, \tau_n$ are positive constants. Three numerical examples have been considered to demonstrate the adaptability of RKHaM to DDEs. This paper has been organized as follows:

In the following Section, the fourth order RKAM and RKHaM formula for solving ODEs have been briefly mentioned. Then the adaptability of RKHaM formula for solving DDEs has been discussed with the convergence and stability for RKHaM. Finally, three numerical examples have been provided to illustrate the effectiveness of RKHaM.

RKHaM METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

Consider the first order equation of the form

$$y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0.$$

The fourth order RKAM formula of the (n+1)th increment in y is computed as

$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right]$$

$$= y_n + \frac{h}{3}\sum_{i=1}^{3}\frac{k_i + k_{i+1}}{2} = y_n + \frac{h}{3}\sum_{i=1}^{3}(AM)$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hk_1\right)$$
$$k_3 = f\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hk_2\right)$$
$$k_4 = f(x_n + h,\ y_n + hk_3)$$

The Harmonic mean of two points y_1 and y_2 is defined as

$$\text{HaM} = \frac{y_1 y_2}{y_1 + y_2}.$$

By replacing the AM in eqn. (2.1) by HaM, the fourth order RKHaM formula is written as

$$y_{n+1} = y_n + \frac{2}{3}h\left[\frac{k_1k_2}{k_1+k_2} + \frac{k_2k_3}{k_2+k_3} + \frac{k_3k_4}{k_3+k_4}\right]$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$
$$k_3 = f\left(x_n + \frac{1}{2}h, y_n - \frac{1}{8}hk_1 + \frac{5}{8}hk_2\right)$$
$$k_4 = f\left(x_n + h, y_n - \frac{1}{4}hk_1 + \frac{7}{20}hk_2 + \frac{9}{10}hk_3\right)$$

The Butcher array for RKHaM is

0			
1/2	1/2		
1/2	-1/8	5/8	
1	-1/4	7/20	9/10
	1/3	1/3	1/3

RKHaM METHOD FOR DELAY DIFFERENTIAL EQUATIONS

Consider the first order DDE of the form

$$y'(t) = f(t, y(t), y(t-\tau)),\quad t > t_0$$
$$y(t) = \Phi(t),\quad t \le t_0$$

where $\Phi(t)$ is the initial function and τ is the delay term. When we adapt the fourth order RKHaM formula to DDE, we have

$$y_{n+1} = y_n + \frac{2}{3}h\left[\frac{k_1k_2}{k_1+k_2} + \frac{k_2k_3}{k_2+k_3} + \frac{k_3k_4}{k_3+k_4}\right]$$

where

$$k_i = f\left(x_n + c_ih, y_n + h\sum_{j=1}^{4} a_{ij}k_j, y(x_n + c_ih - \tau)\right),\ (\text{i} = 1,\ldots,4).$$

The above RKHaM formula can also be extended to solve DDEs with multiple delays.

CONVERGENCE AND STABILITY OF RKHaM

Order of Convergence of RKHaM:

When we use RKHaM formula to solve DDEs, the delay term $y(x_n + c_ih - \tau)$ need to be interpolated to approximate the value. Many techniques are available for the approximation. In this paper linear and Lagrange interpolation are used to approximate the delay term. The interpolation have to be adapted to the order of the method.

For any given RK method, its adaptation to DDEs by means of interpolation procedure has an order of convergence equal to $\min\{p, q\}$ where p denotes the order of consistency of the RK method and q is the number of support points of the interpolation procedure. (See [11])

For linear interpolation we are using two support points so that the order of convergence of RK method is two (See [12]) which is less than the order of the RK method. For Lagrange interpolation we are using five support points so that the order of convergence of RK method is four for DDEs also.

Linear stability for RKHaM:

There are many concepts of stability of numerical methods when applied to DDE, depending on the test equation as well as the delay term involved. Here our attention is to a linear test equation with a constant delay τ,

$$y'(x) = \lambda y(x) + \mu y(x-\tau),\ t \ge t_0$$
$$y(t) = \Phi(t),\quad t \le t_0$$

where $\lambda, \mu \in R$, $\tau > 0$ and Φ is continuous.

The stability function of s-stage RK method for ODE is given by

$$r(\text{z}) = 1 + \text{zb}^T (I - zA)^{-1} e,$$

where

$$z = h\lambda,\ e = (1,1,\ldots,1)^T,\ b = (b_1, b_2, \ldots b_s)^T,\ A = (a_{ij}).$$

In the case of DDEs, we consider $\tau = Nh$ where N is a positive integer. In this case when $u(nh + c_i h - \tau)$ is required a previous internal stage-value $Y_{n-N,i}$ is used. Here we consider the stability properties of a recurrence of the form

$$y_{n+1} = \left[1 + \lambda h b^T (I - \lambda hA)^{-1} e\right] y_n + \mu h b^T (I - \lambda hA)^{-1} u_{n-N}$$

where u_{n-N} is a vector consisting of 'back- values' $u(nh + c_i h - \tau)$.

This can be expressed in a convenient form as

$$y_{n+1} = r(\lambda h) y_n + \mu h b^T (\mathrm{I} - \lambda hA)^{-1} u_{n-N}.$$

If we write $S \equiv S(\lambda h) = (I - \lambda hA)^{-1}$ and $\Phi_n = \left[Y_{n,1}, ..., Y_{n,\nu}\right]^T$, the above can be written as

$$y_{n+1} = r(\lambda h) y_n + \mu h b^T S \phi_{n\_N}$$

We can express this as the recurrence:

$$\Phi_n = X\Phi_{n-1} + Z\Phi_{n-N}$$

where

$$X = \left(\begin{array}{c|c} r(\lambda h) & 0 \\ Se & 0 \end{array}\right) \text{ and } Z = \left(\begin{array}{c|c} 0 & \mu h b^T S \\ 0 & \mu h SA \end{array}\right).$$

This recurrence is stable if the zeros ς_i of the stability polynomial

$$S_h(\lambda, \mu; \varsigma) = \det\left[\varsigma^N I - \varsigma^{N-1} X - Z\right].$$

The root condition for stability is the requirement that all the zeros ς_i of $S_h(\lambda, \mu; \varsigma)$ satisfy $|\varsigma_i| \le 1$, and if $|\varsigma_i| = 1$ then ς_i is semi-simple.

The stability polynomial of RKAM is obtained as,

$$S(\lambda h, \mu h; \varsigma) = \varsigma^{5N} - \varsigma^{5N-1}\left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24}\right)$$
$$-\varsigma^{4N-1}(\mu h)\left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6}\right)$$
$$-\varsigma^{3N-1}\frac{(\mu h)^2}{2}\left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right) - \varsigma^{2N-1}\frac{(\mu h)^3}{6}(\lambda h) - \varsigma^{N-1}\left(\frac{\mu h}{24}\right)^4$$

and the stability polynomial of RKHaM is obtained as,

$$S(\lambda h, \mu h; \varsigma) = \varsigma^{5N} - \varsigma^{5N-1}\left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} - \frac{3(\lambda h)^5}{128}\right)$$
$$-\varsigma^{4N-1}(\mu h)\left(1 + \frac{2}{3}\lambda h + \frac{5}{16}(\lambda h)^2\right)$$
$$-\varsigma^{3N-1}(\mu h)^2\left(\frac{1}{3} + \frac{5}{16}(\lambda h)\right) - \varsigma^{2N-1}(\mu h)^3\left(\frac{5}{48}\right).$$

NUMERICAL EXAMPLES

Example 1

Consider the first order DDE with single delay

$$y'(t) = y(t-1)$$
$$y(t) = \lambda e^t, \quad -1 \le t \le 0$$

with exact solution is,

$$y(t) = \begin{cases} \lambda + \dfrac{(-1 + e^t)\lambda}{e} & \text{on } [0,1] \\ \lambda + \dfrac{\lambda}{e} + e^{(-2+t)}\lambda + t\lambda - \dfrac{t\lambda}{e} & \text{on } [1,2] \\ 2\lambda - \dfrac{2\lambda}{e} + e^{(-3+t)}\lambda + \dfrac{t^2\lambda}{2} - \dfrac{t^2\lambda}{2e} & \text{on } [2,3] \end{cases}$$

This problem is solved by RKAM and RKHaM by using linear interpolation and Lagrange interpolation for the delay term with $\lambda = 1$. The absolute error results are shown in the following Tables 1 and 2.

Table 1
Results of Example 1 (Linear Interpolation)

Time	Absolute Error (RKAM)	Absolute Error (RKHaM)
0.50	1.99e-006	9.44e-006
1.00	5.27e-006	2.63e-006
1.50	7.71e-006	4.42e-006
2.00	1.27e-005	7.66e-005
2.50	1.79e-005	1.00e-005
3.00	2.61e-005	1.46e-005

Table 2
Results of Example 1 (Lagrange Interpolation)

Time	Absolute Error	Absolute Error

	(RKAM)	(RKHaM)
0.50	3.89e-013	3.98e-012
1.00	5.27e-006	1.05e-010
1.50	5.27e-006	7.90e-010
2.00	5.27e-006	7.59e-010
2.50	7.90e-006	1.05e-010
3.00	1.05e-005	4.13e-010

Example 2:

Consider the system of first order DDE with single delay

$$y'_1(t) = y_2(t)$$

$$y'_2(t) = y_3(t)$$

$$y'_3(t) = -y_1(t) - y_1(t-0.3) + e^{(-t+0.3)}, \qquad 0 \le t \le 1$$

with initial functions

$$y_1(t) = e^{-t}; \quad y_2(t) = -e^{-t}; \quad y_3(t) = e^{-t}, \qquad t \le 0$$

with exact solutions

$$y_1(t) = e^{-t}; \quad y_2(t) = -e^{-t}; \quad y_3(t) = e^{-t}$$

This problem is solved by RKAM and RKHaM by using linear interpolation and Lagrange interpolation for the delay term. The absolute error results are shown in the following Tables 3 - 6.

Table 3

Results of Example 2 (Linear Interpolation)

Time	Absolute Error (RKAM)		
	y_1	y_2	y_3
0.20	3.57e-009	5.27e-008	5.10e-007
0.40	2.72e-008	1.98e-007	9.24e-007
0.60	8.75e-008	4.17e-007	1.25e-006
0.80	1.98e-007	6.94e-007	1.50e-006
1.00	3.68e-007	1.01e-006	1.66e-006

Table 4

Results of Example 2 (Linear Interpolation)

Time	Absolute Error (RKHaM)		
	y_1	y_2	y_3
0.20	3.60e-009	5.30e-008	5.10e-007
0.40	2.73e-008	1.98e-007	9.24e-007
0.60	8.70e-008	4.18e-007	1.25e-006
0.80	1.98e-007	6.95e-007	1.50e-006
1.00	3.68e-007	1.01e-006	1.66e-006

Table 5

Results of Example 2 (Lagrange Interpolation)

Time	Absolute Error (RKAM)		
	y_1	y_2	y_3
0.20	8.57e-013	8.58e-013	8.43e-013
0.40	1.40e-012	1.41e-012	1.35e-012
0.60	1.72e-012	1.75e-012	1.50e-012
0.80	1.86e-012	1.98e-012	1.37e-012
1.00	1.87e-012	2.16e-012	1.07e-12

Table 6

Results of Example 2 (Lagrange Interpolation)

Time	Absolute Error (RKHaM)		
	y_1	y_2	y_3
0.20	3.44e-015	3.44e-015	3.44e-015
0.40	7.00e-015	7.00e-015	7.00e-015
0.60	1.90e-014	1.89e-014	1.92e-014
0.80	2.58e-014	2.57e-014	2.49e-014
1.00	2.90e-014	2.91e-014	2.93e-014

Example 3:

Consider the system of first order DDE with multiple delays

$$y'_1(t) = y_5(t-1) + y_3(t-1); \qquad y'_2(t) = y_1(t-1) + y_2(t-\tfrac{1}{2});$$

$$y'_3(t) = y_3(t-1) + y_1(t-\tfrac{1}{2}); \qquad y'_4(t) = y_5(t-1)y_4(t-1);$$

$$y'_5(t) = y_1(t-1) \qquad \text{for } t \ge 0$$

with initial functions

$$y_1(t) = e^{(t+1)}; \quad y_2(t) = e^{(t+\frac{1}{2})}; \quad y_3(t) = \sin(t+1);$$

$$y_4(t) = e^{(t+1)}; \quad y_5(t) = e^{(t+1)} \qquad \text{for } t \le 0.$$

with analytical solutions

$$y_1(t) = e^t - \cos t + e; \quad y_2(t) = 2e^t + e^{\frac{1}{2}} - 2;$$

$$y_3(t) = e^{(t+\frac{1}{2})} - \cos t + 1 - e^{\frac{1}{2}} + \sin(1);$$

$$y_4(t) = \frac{1}{2}e^{2t} - \frac{1}{2} + e; \quad y_5(t) = e^t + e - 1 \quad \text{for } 0 \le t \le \frac{1}{2}.$$

This problem is solved by RKAM and RKHaM by using linear interpolation and Lagrange interpolation for the delay term. The absolute error results are shown in the following Tables 7 and 8.

Table 7

Results of Example 3 (Linear Interpolation)

Variables	Absolute Error (RKAM) at t=0.5	Absolute Error (RKHaM) at t=0.5
y_1	4.39e-006	2.64e-006
y_2	1.08e-006	5.40e-006
y_3	7.89e-006	5.81e-007
y_4	1.43e-006	1.37e-011
y_5	5.41e-006	2.70e-006

Table 8
Results of Example 3 (Lagrange Interpolation)

Variables	Absolute Error (RKAM) at t=0.5	Absolute Error (RKHaM) at t=0.5
y_1	7.07e-012	1.76e-010
y_2	2.93e-008	2.08e-010
y_3	1.43e-007	1.46e-010
y_4	7.16e-011	3.58e-010
y_5	2.70e-011	6.75e-011

CONCLUSION

In this paper, RKHaM formula has been adopted to solve the delay differential equations with constant lags. The effectiveness of this approach has been illustrated via examples of DDE with single delay and multiple delays. To interpolate the delay term, both linear interpolation and Lagrange interpolation have been considered here.

The numerical outcomes also have been compared with RK method based on arithmetic mean. From the numerical results, it is observed that the RKHaM method is well suitable for solving DDEs. It also suggests that the best results can be obtained when we use Lagrange interpolation with suitable number of support points for getting fourth order convergence.

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