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Original Research Paper

Bayesian Inference on A Mixture of Geometric and Degenerate Distribution: A Special Case of Zero Inflated Geometric Distribution, and Posterior Odds Ratio

STATISTICS

KEYWORDS : Geometric Distribution, Zero Inflation, Degenerate Distribution, Informative Priors, Non-Informative Priors, Power Series Distribution (PSD), Posterior Odds Ratio (POR).

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ABSTRACT

Here, we have developed a change point model by considering the mixture of two distributions. Let us consider that a random sequence of X1, X2, X3,. . . ,Xm was observed from the Zero Inflated Geometric Distribution with proportion p_1 and θ_1. Later it was found that there was a change in the system at some unknown point of time 'm' ($m<n$) and it was reflected in the sequence after Xm with proportion p_2 and θ_2. Then we have obtained the Posterior Odds Ratio (POR) of the change point 'm' under beta priors and also under non-informative prior. In the next section, we have obtained posterior densities of 'm' for p_1 and p_2 known using beta prior under H_1 and H_0 respectively. Then, we have obtained the marginal posterior densities of 'm' for p_1 and p_2 known using non-informative prior of θ_1 and θ_2 under H_1 and H_0 respectively. Later, numerical study is done on the generated observations. In the last section, we have studied the sensitivity of Posterior Odds Ratio with respect to change in prior of the parameters.

1. INTRODUCTION

One-parameter discrete exponential families are suitable for modelling count data. Power series distributions form a useful subclass of it. A zero-inflated power series distribution is a mixture of a power series distribution and a degenerate distribution at zero, with a mixing probability 'p' for the degenerate distribution. This distribution is useful for modelling count data that may have extra zeros. A question arises whether the mixture model can be reduced to the power series portion corresponding to p = 0, or whether there are so many zeros in the data that zero inflation related to the pure power series distribution be included in the model, i.e. ; $p \geq 0$. The problem is partially difficult as p = 0 is a boundary point. A Bayesian test for the problem based on recognizing that the parameter space can be expanded to allow *p* to be negative was presented by a Statistician named **A. Bhattacharya** and others in the year **2008**. According to that, using a posterior probability as a test statistic has slightly higher power on the most important ranges of the sample size *n* and parameter values than the score test and likelihood ratio test in simulations. The method which we are going to use also performs well on three real data sets.

Models for count data often fail to fit in practice. It happens because of the presence of more zeros in the data than explained by a standard model. This situation is often called zero inflation. It is so because the number of zeros is inflated from the baseline number of zeros that would be expected in, say, a one-parameter discrete exponential family. Zero inflation is a special case of over dispersion. It contradicts the relationship between the mean and variance in a one-parameter exponential family. One way to address this is to use a two-parameter distribution so that the extra parameter permits a larger variance.

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Johnson, Kotz and **Kemp** have discussed a simple modification of a **Power Series Distribution (PSD)** f (*|θ) to handle extra zeros. An extra proportion of zeros, p, is added to the proportion of zeros from the original discrete distribution, while decreasing the remaining proportions in an appropriate way. So the zero inflated PSD is defined as

$$f^*(y|p,\theta) = \begin{cases} p(1-p)\mathrm{f}(0|\theta) & , \ y = 0 \\ (1-p)\mathrm{f}(\mathrm{y}|\theta) & , \ y \geqslant 0 \end{cases}$$

where $\theta \in \Theta$, the parameter space and the maxing parameter p ranges over the interval,

$$-\frac{\mathrm{f}(0|\theta)}{(1-\mathrm{f}(0|\theta))} < p < 1$$

This allows the distribution to be well defined for certain negative values of p which depends on θ. The mixing interpretation is lost when $p < 0$. These values have a natural interpretation in terms of zero-deflation, relative to a PS model. Correspondingly, p > 0 can be regarded as zero inflation relative to a PS model. Here, we note that the PS family contains all discrete one-parameter exponential families. So an appropriate choice of PS model permits any desired interpretation for the data corresponding to the second term. The first term allows an extra proportion p of zeros to be added to the discrete PSD. This data is effectively regarded as a sort of contamination.

$$f^*(x|p,\theta) = \begin{cases} p(1-p)(1-\theta) & , \ x = 0 \\ (1-p)(1-\theta)\theta^x & , \ x \geqslant 0 \end{cases} \qquad (1)$$

Further, we shall consider that a random sequence of $X_1, X_2, X_3, \ldots, X_m$ was observed from the Zero Inflated Geometric Distribution with proportion p_1 and θ_1. Later it was found that there was a change in the system at some unknown point of time 'm' (m<n) and it was reflected in the sequence after X_m with proportion p_2 and θ_2. Then we have obtained the **Posterior Odds Ratio (POR)** of the change point **'m'** under beta priors and also under non-informative prior. In the next section, we have obtained posterior densities of 'm' for p_1 and p_2 known using beta prior under H_1 and H_0 respectively. Then, we have obtained the marginal posterior densities of 'm' for p_1 and p_2 known using non-informative prior of θ_1 and θ_2 under H_1 and H_0 respectively. Later, numerical study is done on the generated observations. In the last section, we have studied the sensitivity of Posterior Odds Ratio with respect to change in prior of the parameters.

2. PROPOSED CHANGE POINT MODEL

Let $X_1, X_2, \ldots, X_n$ ($n \geq 3$) be a sequence of observed count data. Let first 'm' observations $X_1, X_2, \ldots, X_m$ have come from the Zero Inflated Geometric distribution with probability mass function (p_1, θ_1) and later (n-m) observations coming from the Zero Inflated Geometric distribution with probability mass function (p_2, θ_2). Hence, we proposed following change point model.

$$f^*(x_i|p_1,\theta_1) = (p_1 + (1-p_1)(1-\theta_1))^{I(x_i)} \; [(1-p_1)(1-\theta_1)]^{1-I(x_i)} \theta_1^{\sum_{i=1}^{m} x_i(1-I(x_i))}$$

$$f^*(x_i|p_2,\theta_2) = \left(p_2 + (1-p_2)(1-\theta_2)\right)^{I(x_i)}$$

$$((1-p_2)(1-\theta_2){\theta_2}^{x})^{1-I(x_i)} \theta_2^{\sum_{i=m+1}^{n} x_i \, [1-I(x_i)]} \qquad (2)$$

where, $I(x_i) = \begin{cases} 1, & x_i = 0 \\ 0, & x_i > 0 \end{cases}$

'm' is the change point while p_1 and p_2 are proportions.

$$\sum_{i=m+1}^{n} \left(1 - I(x_i)\right) = n - m - \mathrm{d_n} + \mathrm{d_m}$$

$$\sum_{i=1}^{m} x_i(1 - \mathrm{I(x_i)}) = \mathrm{S_m}$$

$$\sum_{i=m+1}^{n} x_i(1 - \mathrm{I(x_i)}) = \mathrm{S_n} - \mathrm{S_m} \tag{3}$$

3. POSTERIOR ODDS RATIO

Here, we note that if m=n, then there is no change in the specified model and if $1 \leq m \leq n - m$, then there is a single change point in the model. Thus, the presence of structural stability (instability) in the system may be decided by testing the null hypothesis as under:

H0 : m = n

with that of the alternative hypothesis of exactly one change, ie ;

H1 : $1 \leq m \leq n - m$

For that, we consider the conditional prior pmf of 'm' to be as under:

$$g(m|p) = \begin{cases} p & , \ m = 0 \\ (1-p)/n - 1 & , \ 1 \leq m \leq n - 1 \\ 0 & , otherwise \end{cases} \tag{4}$$

where, 0<p<1.

Further, we assume that the marginal prior pdf of *'p'* be Beta Type-I, ie; $\beta(a_3, b_3)$; $a_3, b_3 > 0$. Then the marginal prior of 'm' is given by,

$$g(m) = \int_0^1 g(m/p)g(p)dp = \frac{1}{\beta(a_3, b_3)} \int_0^1 p^{(a_3+1)-1} (1-p)^{b_3-1} dp \qquad , if\ m = n$$

$$= \frac{\beta(a_3+1, b_3)}{\beta(a_3, b_3)}$$

$$= \frac{a_3}{(a_3 + b_3)}$$

Similarly, $g(m) = \frac{1}{\beta(a_3,\beta_3)(m-1)} \int_0^1 p^{a_3-1} (1-p)^{b_3-1} dp\ , if\ 1 \le m \le n-1$

$$= \frac{\beta(a_3, b_3)}{\beta(a_3, b_3)(m-1)}$$

$$= \frac{b_3}{(a_3+b_3)(n-1)}$$

and,

$g(m) = 0\ , otherwise$

Thus, we have

$$g(m) = \begin{cases} \frac{a_3}{(a_3 + b_3)} & , m = n \\ \frac{b_3}{(a_3+b_3)(n-1)} & ,\ 1 \le m \le n-1 \\ 0 & , otherwise \end{cases} \qquad (5)$$

The likelihood functions corresponding to H_0 & H_1 are given by $L_0(\theta_1, p_1 | \underline{X})$ and $L_1(\theta_1, \theta_2, p_1, p_2, m\ | \underline{X})$ respectively.

The likelihood function of the parameters $\theta_1, \theta_2, p_1, p_2\ and\ m$ given the sample information $\underline{X} = \{X_1\ ,\ X_2\ , \ldots, X_m, X_{m+1}\ , \ldots, X_n\}$ is given by,

$$L_1(\theta_1,\theta_2,p_1,p_2,m|\underline{X}) = (p_1+(1-p_1)(1-\theta_1))^{\sum_{i=1}^{m} I(x_i)}$$

$$((1-p_1)(1-\theta_1))^{\sum_{i=1}^{m}(1-I(x_i))}\theta_1^{\sum_{i=1}^{m} x_i(1-\mathrm{I}(\mathrm{x_i}))}(p_2+(1-p_2)(1-\theta_2))^{\sum_{i=m+1}^{n} I(x_i)}$$

$$((1-p_2)(1-\theta_2)\theta_2{}^{x})^{\sum_{i=m+1}^{n}(1-I(x_i))}\theta_2^{\sum_{i=m+1}^{n} x_i(1-\mathrm{I}(\mathrm{x_i})}$$

$$= (p_1+(1-p_1)(1-\theta_1))^{\sum_{i=1}^{m} I(x_i)}.(1-\theta_1)^{\sum_{i=1}^{m}(1-I(x_i))}(1-p_1)^{\sum_{i=1}^{m}(1-I(x_i))}.\theta_1^{\sum_{i=1}^{m} x_i(1-\mathrm{I}(\mathrm{x_i}))}$$

$$(p_2+(1-p_2)(1-\theta_2))^{\sum_{i=m+1}^{n} I(x_i)}.(1-\theta_2)^{\sum_{i=m+1}^{n}(1-I(x_i))}$$

$$(1-p_2)^{\sum_{i=m+1}^{n}(1-I(x_i))}.\theta_2^{\sum_{i=m+1}^{n} x_{i(1-\mathrm{I}(\mathrm{x_i})}}$$

$$= (p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_m}}.(1-\theta_1)^{m-\mathrm{d_m}}.(1-p_1)^{m-\mathrm{d_m}}\ \theta_1{}^{\mathrm{S_m}}$$

$$(p_2+(1-p_2)(1-\theta_2))^{\mathrm{d_n}-\mathrm{d_m}}.(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}}.(1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}}\ \theta_2{}^{\mathrm{S_n}-\mathrm{S_m}} \quad (6)$$

Also, we have

$$L_0(\theta_1,\mathrm{p}_1,m|\underline{X}) = (p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_n}}.(1-\theta_1)^{n-\mathrm{d_n}}.(1-p_1)^{n-\mathrm{d_n}}.\theta_1{}^{\mathrm{S_n}} \quad (7)$$

where, we have,

$$\sum_{i=1}^{m} I(x_i) = \mathrm{d_m}$$

$$\sum_{i=1}^{m}\big(1-I(x_i)\big) = m-\mathrm{d_m}$$

$$\sum_{i=m+1}^{n} I(x_i) = \mathrm{d_n}-\mathrm{d_m}$$

Posterior Odds Ratio (POR) in favour of hypothesis of no change in the model against the alternative of one change works out to be as under:

$$P^*_{01} = P_r(m = n \mid \underline{x}) / 1 - P_r\{m = n / \underline{X}\} \quad (8)$$

In the following sub-sections, we have computed the POR in favour of $\boldsymbol{H_0}$ when

(i) Prior Distributions are **Beta Type-I**.

(ii) Prior Distributions of $\theta_1, \theta_2, p_1, p_2$ and m are **Non-Informative**.

We also examine the robustness of the test when prior distributions are Beta Type-I.

4. MARGINAL POSTERIOR DENSITY OF X USING BETA PRIORS

The ML methods as well as other classical approaches are based only on the empirical information which the data provides. However, Bayes procedure seems to be an attractive inferential method when we have some technical knowledge on the parameters of the distribution available. The Bayes procedure is based on a posterior density, which is proportional to the product of the likelihood function and a prior joint density, representing uncertainty on the parameters values.

Let the marginal prior distribution of θ_1 and θ_2 be following Beta Distribution with mean $\mu_i \ (i = 1,2)$ and standard deviation to be $\sigma_i \ (i = 1,2)$.

So, $g_1(\theta_i) = \frac{(1-\theta_i)^{a_i - 1} {\theta_i}^{b_i - 1}}{\beta(a_i, b_i)}$ where $a_i, b_i > 0$, $i = 1,2$

Having $\mu_i = \frac{a_i}{a_i + b_i}$, $\sigma_1 = \frac{a_i b_i}{[(a_i+b_i)^2(a_i+b_i+1)]}$ which gives

$$a_i = \sigma_i^{-1} \left[(1-\mu_i)\mu_i^2 - \mu_i \sigma_i\right]$$

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$b_i = \mu_i^{-1}\,(1-\mu_i)a_i$ where $i = 1,2$ (9)

Let the marginal prior distribution of $p_1\; and\; p_2$ be following Beta Distribution with mean $\mu_i\;(i = 3,4)$ and standard deviation to be $\sigma_i\;(i = 3,4)$

So, $g_1(p_j) = \frac{(1-p_j)^{c_j-1}{p_j}^{d_j-1}}{\beta(c_j,d_j)}$ where $c_j\,, d_j > 0\,, j = 1,2$

Having $\mu_i = \frac{c_j}{c_j + d_j}\,, \sigma_i = \frac{c_j d_j}{[(c_j+d_j)^2(c_j+d_j+1)]}$ which gives

$c_j = \sigma_i^{-1}\,[(1-\mu_i)\mu_i^2 - \mu_i\sigma_i]$ where $i = 3,4$

$d_j = \mu_i^{-1}\,(1-\mu_i)c_j$ where $j = 1,2$ (10)

We assume that $\theta_1, \theta_2, p_1, p_2$ and m are priori independent. The joint prior density under $\boldsymbol{H_1}$ is as under:

$$g_1(\theta_1, \theta_2, p_1, p_2, m)$$

$$= \frac{b_3}{(a_3+b_3)} \frac{(1-\theta_1)^{a_1-1}{\theta_1}^{b_1-1}(1-\theta_2)^{a_2-1}{\theta_2}^{b_2-1}(1-p_1)^{d_1-1}{p_1}^{c_1-1}(1-p_2)^{d_2-1}{p_2}^{c_2-1}}{(n-1)\beta(a_1,b_1)\beta(a_2,b_2)\beta(c_1,d_1)\beta(c_2,d_2)}$$

$$= k(1-\theta_1)^{a_1-1}{\theta_1}^{b_1-1}(1-\theta_2)^{a_2-1}{\theta_2}^{b_2-1}(1-p_1)^{d_1-1}{p_1}^{c_1-1}$$

$$(1-p_2)^{d_2-1}{p_2}^{c_2-1}(1-p_2)^{c_2-1}{p_2}^{d_2-1} \quad (11)$$

where, $k = \frac{b_3}{(a_3+b_3)} \frac{1}{(n-1)\beta(a_1,b_1)\beta(a_2,b_2)\beta(c_1,d_1)\beta(c_2,d_2)}$ (12)

The joint posterior density of parameters $\theta_1, \theta_2, p_1, p_2$ and m is obtained using the likelihood function and the joint prior density as under:

$$g_1(\theta_1, \theta_2, p_1, p_2, m \mid \underline{X}) = \frac{L_1(\theta_1, \theta_2, p_1, p_2, m|\underline{X})\;\; g_1(\theta_1, \theta_2, p_1, p_2, m)}{h_2(\underline{X})}$$

$L_1(\theta_1,\theta_2,p_1,p_2,m\,|\underline{\mathrm{X}})\;\; g_1(\theta_1,\theta_2,p_1,p_2,m) =$

$(p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_m}}.(1-\theta_1)^{m-\mathrm{d_m}}.(1-p_1)^{m-\mathrm{d_m}}.\theta_1{}^{\mathrm{S_m}}$

$(p_2+(1-p_2)(1-\theta_2))^{\mathrm{d_n}-\mathrm{d_m}}.(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}}.(1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}}.\theta_2{}^{\mathrm{S_n}-\mathrm{S_m}}$

$.\,k(1-\theta_1)^{a_1-1}\theta_1{}^{b_1-1}(1-\theta_2)^{a_2-1}\theta_2{}^{b_2-1}(1-p_1)^{d_1-1}p_1{}^{c_1-1}$

$(1-p_2)^{d_2-1}p_2{}^{c_2-1}(1-p_2)^{c_2-1}p_2{}^{d_2-1}$

$= (p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_m}}(1-p_1)^{m-\mathrm{d_m}+d_1-1}p_1{}^{c_1-1}$

$(1-\theta_1)^{m-\mathrm{d_m}+a_1-1}\theta_1{}^{\mathrm{S_m}+b_1-1}.(p_2+(1-p_2)(1-\theta_2))^{\mathrm{d_n}-\mathrm{d_m}}$

$(1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+d_2-1}p_2{}^{c_2-1}(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+a_2-1}\theta_2{}^{\mathrm{S_n}-\mathrm{S_m}+b_2-1}$ (13)

Then the marginal density of X will be $h_2(\underline{X}) = h_0(\underline{X}) + h_1(\underline{X})$.

and $h_1(\underline{\mathrm{X}})$ is the marginal posterior density of X, under $\boldsymbol{H_1}$.

$$h_1(\underline{\mathrm{X}}) = \sum_{m=1}^{n-1}\int_0^1\int_0^1\int_0^1\int_0^1 L_1(\theta_1,\theta_2,p_1,p_2,m|\underline{\mathrm{X}})\;\; g_1(\theta_1,\theta_2,p_1,p_2,m)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}p_1\,\mathrm{d}p_2$$

$= k\sum_{m=1}^{n-1}\int_0^1\{\int_0^1(p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_m}}(1-p_1)^{m-\mathrm{d_m}+d_1-1}p_1{}^{c_1-1}\,\mathrm{d}p_1\}$

$(1-\theta_1)^{m-\mathrm{d_m}+a_1-1}\theta_1{}^{\mathrm{S_m}+b_1-1}\,\mathrm{d}\theta_1$

$\int_0^1\{\int_0^1(p_2+(1-p_2)(1-\theta_2))^{\mathrm{d_n}-\mathrm{d_m}}(1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+d_2-1}p_2{}^{c_2-1}\,\mathrm{d}p_2\}.$

$(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+a_2-1}\theta_2{}^{\mathrm{S_n}-\mathrm{S_m}+b_2-1}\,\mathrm{d}\theta_2$

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$$= k \sum_{m=1}^{n-1} I_1(\mathrm{m}) I_2(\mathrm{m}) \quad (14)$$

where,

$$I_1(\mathrm{m}) = \int_0^1 \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}+d_1-1} {p_1}^{c_1-1}\, \mathrm{d}p_1\}$$

$$(1-\theta_1)^{m-\mathrm{d_m}+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1}\, \mathrm{d}\theta_1$$

$$= \int_0^1 (1-\theta_1)^{m+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1} \Gamma \mathrm{c}_1 \Gamma(m - \mathrm{d_m} + d_1)$$

$${}_2F_1[c_1, -\mathrm{d_m}, c_1 + m - \mathrm{d_m} + d_1, \frac{\theta_1}{\theta_1 - 1}]\, \mathrm{d}\theta_1$$

$$= \Gamma \mathrm{c}_1 \Gamma(m - \mathrm{d_m} + d_1) \int_0^1 (1-\theta_1)^{m+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1}$$

$${}_2F_1[c_1, -\mathrm{d_m}, c_1 + m - \mathrm{d_m} + d_1, \frac{\theta_1}{\theta_1 - 1}]\, \mathrm{d}\theta_1 \quad (15)$$

where ${}_2F_1\left[c_1, -\mathrm{d_m}, c_1 + m - \mathrm{d_m} + d_1, \frac{\theta_1}{\theta_1 - 1}\right]$ is a hyper geometric function defined below.

The Gauss hyper geometric function in three parameter a, b and c denoted by ${}_2F_1$, is defined by

${}_2F_1(a,b,c;x) = \sum_{m=0}^{\infty} \frac{(a,m)(b,m)x^m}{(c,m)m!}$ $\quad for\ |x| < 1$ with Pochhamer coefficients.

$$(a,m) = \frac{\Gamma(a+m)}{\Gamma(a)} \qquad for\ m \geq 1\ and\ (a,0) = 1$$

This function can be integrally represented as

$${}_2F_1(a,b,c;x) = \int_0^1 \frac{[\,u^{a-1}\,(1-u)^{c-a-1}\,(1-xu)^{-b}]}{B(a, c-a)}\, du$$

The symbols Γ and B (*) denoting the usual functions Gamma and Beta respectively. This function is a solution to a hyper geometric differential equation. It is known as Gauss series or the Kummer series.

For the integral $I_2(m)$,

$$I_2(\mathrm{m}) = \int_0^1 \{\int_0^1 (p_2 + (1-p_2)(1-\theta_2))^{\mathrm{d_n - d_m}} (1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+d_2-1} {p_2}^{c_2-1}\, \mathrm{d}p_2\}$$

$$(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+a_2-1} {\theta_2}^{\mathrm{S_n - S_m}+b_2-1}\, \mathrm{d}\theta_2$$

$$= \int_0^1 (1-\theta_2)^{n-m+a_2-1} {\theta_2}^{\mathrm{S_n - S_m}+b_2-1}\ \Gamma \mathrm{c}_2 \Gamma(n-m-\mathrm{d_n}+\mathrm{d_m}+d_2)$$

$${}_2F_1[c_2, -(\mathrm{d_n}-\mathrm{d_m}), c_2+n-m-\mathrm{d_n}+\mathrm{d_m}+d_2, \tfrac{\theta_2}{\theta_2-1}]\, \mathrm{d}\theta_2 \qquad (16)$$

where ${}_2F_1[c_2, -(\mathrm{d_n}-\mathrm{d_m}), c_2+n-m-\mathrm{d_n}+\mathrm{d_m}+d_2, \tfrac{\theta_2}{\theta_2-1}]$ is a hyper geometric function as explained earlier.

$$g_0(\theta_1, p_1) = \frac{a_3}{(a_3 + b_3)} \frac{(1-\theta_1)^{a_1-1} {\theta_1}^{b_1-1} (1-p_1)^{d_1-1} {p_1}^{c_1-1}}{\beta(a_1,b_1)\beta(c_1,d_1)}$$

$h_0(\underline{\mathrm{X}})$ is the marginal posterior density of $\underline{\mathrm{X}}$ under $\boldsymbol{H_0}$.

$$h_0(\underline{\mathrm{X}}) = \int_0^1 \int_0^1 L_0(\theta_1, p_1, |\underline{\mathrm{X}})\, g_0(\theta_1, p_1)\, \mathrm{d}\theta_1\, \mathrm{d}p_1$$

$$= k_1 \int_0^1 \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}+d_1-1} {p_1}^{c_1-1}\, \mathrm{d}p_1\}$$

$$(1-\theta_1)^{m-\mathrm{d_m}+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1}\, \mathrm{d}\theta_1$$

$$= k_1\, I_3(\mathrm{m}) \qquad (17)$$

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where $k_1 = \frac{a_3}{(a_3+b_3)} \frac{1}{\beta(a_1,b_1)\beta(c_1,d_1)}$ and

$$I_3(\mathrm{m}) = \int_0^1 \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}+d_1-1} {p_1}^{c_1-1}\, \mathrm{d}p_1\}$$

$$(1-\theta_1)^{m-\mathrm{d_m}+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1}\, \mathrm{d}\theta_1$$

$$= \int_0^1 (1-\theta_1)^{\mathrm{m}+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1} \Gamma \mathrm{c}_1 \Gamma(n-\mathrm{d_m}+d_1)$$

$$_2F_1[c_1, -\mathrm{d_m}, c_1+m-\mathrm{d_m}+d_1, \tfrac{\theta_1}{\theta_1-1}]\, \mathrm{d}\theta_1$$

$$= \Gamma \mathrm{c}_1 \Gamma(m-\mathrm{d_m}+d_1) \int_0^1 (1-\theta_1)^{m+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1}$$

$$_2F_1[c_1, -\mathrm{d_m}, c_1+m-\mathrm{d_m}+d_1, \tfrac{\theta_1}{\theta_1-1}]\, \mathrm{d}\theta_1$$

5. MARGINAL POSTERIOR DENSITY OF ‘m’ UNDER $\mathbf{H_1}$ AND BETA PRIOR

Here, we have derived the posterior density of change point ‘m’ of the model under beta prior in the previous section under $\mathbf{H_1}$.

The joint posterior density of θ_1, θ_2 is obtained using the joint posterior density of $\theta_1, \theta_2, p_1, p_2$ *and* m and then integrating with respect to p_1 and p_2 and then taking summation over ‘m’ as under:

$$g_1(\theta_1, \theta_2 | \underline{\mathrm{X}}) = k \sum_{m=1}^{n-1} \int_0^1 \int_0^1 g_1(\theta_1, \theta_2, p_1, p_2, m | \underline{\mathrm{X}})\, \mathrm{d}p_1\, \mathrm{d}p_2$$

$$= k \sum_{m=1}^{n-1} \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}+d_1-1} {p_1}^{c_1-1}\, \mathrm{d}p_1\}$$

$$(1-\theta_1)^{m-\mathrm{d_m}+a_1-1} {\theta_1}^{\mathrm{S_m}+b_1-1}$$

$$\{\int_0^1 (p_2 + (1-p_2)(1-\theta_2))^{\mathrm{d_n}-\mathrm{d_m}} (1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+d_2-1} {p_2}^{c_2-1}\, \mathrm{d}p_2\}$$

$(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}+a_2-1}\theta_2^{\mathrm{S_n}-\mathrm{S_m}+b_2-1}.h_1^{-1}(\underline{\mathrm{X}})$

$=k\sum_{m=1}^{n-1}(1-\theta_1)^{m+a_1-1}\theta_1^{\mathrm{S_m}+b_1-1}(1-\theta_2)^{n-m+a_2-1}\theta_2^{\mathrm{S_n}-\mathrm{S_m}+b_2-1}$

$\Gamma\mathrm{c}_1\Gamma(m-\mathrm{d_m}+d_1)\,{}_2F_1[c_1,-\mathrm{d_m},c_1+m-\mathrm{d_m}+d_1,\frac{\theta_1}{\theta_1-1}]$

$\Gamma\mathrm{c}_2\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m}+d_2)$

${}_2F_1\left[c_2,-(\mathrm{d_n}-\mathrm{d_m}),c_2+n-m-\mathrm{d_n}+\mathrm{d_m}+d_2,\frac{\theta_2}{\theta_2-1}\right]\ h_1^{-1}(\underline{\mathrm{X}})$

$=k\sum_{m=1}^{n-1}\Gamma\mathrm{c}_1\Gamma\mathrm{c}_2\Gamma(m-\mathrm{d_m}+d_1)\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m}+d_2)$

$(1-\theta_1)^{m+a_1-1}\theta_1^{\mathrm{S_m}+b_1-1}\,{}_2F_1[c_1,-\mathrm{d_m},c_1+m-\mathrm{d_m}+d_1,\frac{\theta_1}{\theta_1-1}]$

$(1-\theta_2)^{n-m+a_2-1}\theta_2^{\mathrm{S_n}-\mathrm{S_m}+b_2-1}$

${}_2F_1\left[c_2,-(\mathrm{d_n}-\mathrm{d_m}),c_2+n-m-\mathrm{d_n}+\mathrm{d_m}+d_2,\frac{\theta_2}{\theta_2-1}\right]\ h_1^{-1}(\underline{\mathrm{X}})$ (18)

where ${}_2F_1[c_2,-(\mathrm{d_n}-\mathrm{d_m}),c_2+n-m-\mathrm{d_n}+\mathrm{d_m}+d_2,\frac{\theta_2}{\theta_2-1}]$ is a hyper geometric function as explained earlier.

The marginal density of θ_1 is obtained using the joint posterior density of $\theta_1 and\ \theta_2$ and then integrating with respect to θ_2 as under:

$g_1(\theta_1|\underline{\mathrm{X}})=k\sum_{m=1}^{n-1}\int_0^1 g_1(\theta_1,\theta_2|\underline{\mathrm{X}})\,\mathrm{d}\theta_2$

$=k\sum_{m=1}^{n-1}\Gamma\mathrm{c}_1\Gamma\mathrm{c}_2\Gamma(m-\mathrm{d_m}+d_1)\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m}+d_2)$

$(1-\theta_1)^{m+a_1-1}\theta_1^{\mathrm{S_m}+b_1-1}\,{}_2F_1[c_1,-\mathrm{d_m},c_1+m-\mathrm{d_m}+d_1,\frac{\theta_1}{\theta_1-1}]$

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$$\int_0^1 (1-\theta_2)^{n-m+a_2-1} \theta_2^{\mathrm{S_n}-\mathrm{S_m}+b_2-1}$$

$$_2F_1\left[c_2, -(\mathrm{d_n}-\mathrm{d_m}), c_2+n-m-\mathrm{d_n}+\mathrm{d_m}+d_2, \frac{\theta_2}{\theta_2-1}\right] \mathrm{d}\theta_2 \;\; h_1^{-1}(\underline{\mathrm{X}}) \qquad (19)$$

Marginal posterior density of m is obtained using the joint posterior distribution function,

$$g_1(m\,|\underline{X}) = \mathrm{k}\, I_1(\mathrm{m}) I_2(\mathrm{m})\, h_2^{-1}(\underline{\mathrm{X}}) \qquad (20)$$

6. MARGINAL POSTERIOR DENSITY OF 'm' UNDER $\mathrm{H_0}$ AND BETA PRIOR

Here, we have derived the posterior density of change point 'm' of model under beta prior explained earlier under $\boldsymbol{H_0}$.

The joint posterior density on $\theta_1, \mathrm{p_1}$ and m, say $g_0(\theta_1, \mathrm{p_1}, \mathrm{m}|\underline{\mathrm{X}})$ is calculated using likelihood function and joint prior density under:

$$g_0(\theta_1, \mathrm{p_1}, \mathrm{m}|\underline{\mathrm{X}}) = \frac{a_3}{(a_3+b_3)} \frac{(1-\theta_1)^{a_1-1}\theta_1^{b_1-1}(1-p_1)^{d_1-1}p_1^{c_1-1}}{\beta(a_1,b_1)\beta(c_1,d_1)} (p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_n}}$$

$$(1-\theta_1)^{n-\mathrm{d_n}}.(1-p_1)^{n-\mathrm{d_n}}.\theta_1^{\mathrm{S_n}} \;\; h_2^{-1}(\underline{\mathrm{X}})$$

The marginal density of θ_1 is obtained using the joint posterior density of $\theta_1 and\, p_1$ and then integrating with respect to p_1, we get,

$$g_0(\theta_1, \mathrm{m}|\underline{\mathrm{X}}) = k_1 \sum_{m=1}^{n-1} \Gamma \mathrm{c_1} \Gamma(n-\mathrm{d_n}+d_1)\,(1-\theta_1)^{n+a_1-1}\theta_1^{\mathrm{S_n}+b_1-1}$$

$$_2F_1[c_1, -\mathrm{d_n}, c_1+n-\mathrm{d_n}+d_1, \frac{\theta_1}{\theta_1-1}] h_2^{-1}(\underline{\mathrm{X}})$$

Marginal posterior density of 'm' is obtained using the joint posterior distribution function

$$g_0(m\,|\underline{X}) = k_1\, I_3(m)\, h_2^{-1}(\underline{\mathrm{X}}).$$

where $I_3(m)$ is same as given in (17) and h_2 ($\underline{X}$) is same as given above.

Now, the posterior odds ratio is obtained using equations as under:

$$P_{01}^{*} = \frac{a_3}{b_3}\frac{I_1(m)I_2(m)}{k_1 I_3(m)} \qquad (21)$$

7. POSTERIOR ODDS RATIO UNDER NON-INFORMATIVE PRIOR

In this section, we derive Posterior Odds Ratio (POR) under non-informative prior, ie; Posterior Odds Ratio (POR) in favour of hypothesis of no change in the model against the alternative of one change works out to be.

$$P_{02}^{*} = P_r(m = n \mid \underline{x}) \,/ 1 - P_r\{m = n / \underline{X}\} \qquad (22)$$

Sometimes no prior information or technical knowledge about the parameters is available. Therefore, we take non-informative priors. A non-informative prior is a prior that reflects indifference to all values of the parameter and adds no information to that contained in the empirical data. Thus, a Bayes inference based upon non-informative prior has generally a theoretical interest only. Since from an engineering view point, the Bayes approach is very attractive for it allows incorporating expert opinion or technical knowledge in the estimation procedure. However, such a Bayes inference acquires large interest in solving prediction problems when it is extremely difficult, if at all possible, to find a classical solution for the prediction problem, because classical prediction intervals are numerically equal to the Bayes ones based on the non-informative prior density. Hence, the Bayes approach based on prior ignorance can be viewed as mathematical method for obtaining classical prediction intervals.

Then the joint prior distribution of $\theta_1, \theta_2, p_1, p_2$,m and θ_1, p_1, m under H_1 and H_0 is

$$g_{11}(\theta_1, \theta_2, p_1, p_2, m) = \frac{a_3}{(a_3 + b_3)} \frac{1}{(n-1)(1-\theta_1)\theta_1(1-\theta_2)\theta_2(1-p_1)(1-p_2)}$$

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and $g_{01}(\theta_1, p_1, m) = \frac{b_3}{(a_3 + b_3)} \frac{1}{(1-\theta_1)\theta_1(1-p_1)(1-p_2)}$ respectively.

and also we have $k_3 = \frac{a_3}{(a_3 + b_3)} \frac{1}{(n-1)}$

The joint posterior density of parameters $\theta_1, \theta_2, p_1, p_2$ and m under non-informative prior is obtained using the likelihood function and the joint prior density of the parameters under non-informative prior.

$$g_2(\theta_1, \theta_2, p_1, p_2, m|\underline{X}) = \frac{L_1(\theta_1, \theta_2, p_1, p_2, m|\underline{X})\, g_{11}(\theta_1, \theta_2, p_1, p_2, m)}{h_5(\underline{X})} \quad (23)$$

Then the marginal density of $\underline{X}$ will be $h_5(\underline{X}) = h_3(\underline{X}) + h_4(\underline{X})$.

where, $h_3(\underline{X})$ is the marginal density of $\underline{X}$ under non-informative priors and $\boldsymbol{H_1}$ and $h_4(\underline{X})$ is the marginal density of $\underline{X}$ under non-informative priors and $\boldsymbol{H_0}$.

Here,

$$h_3(\underline{X}) = \sum_{m=1}^{n-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 L_1(\theta_1, \theta_2, p_1, p_2, m|\underline{X}) g_{11}(\theta_1, \theta_2, p_1, p_2, m)\, d\theta_1\, d\theta_2\, dp_1\, dp_2$$

$$= k_3 \sum_{m=1}^{n-1} \int_0^1 \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{d_m} (1-p_1)^{m-d_m-1}\, dp_1\}$$

$$(1-\theta_1)^{m-d_m-1} \theta_1^{S_m-1}\, d\theta_1$$

$$\int_0^1 \{\int_0^1 (p_2 + (1-p_2)(1-\theta_2))^{d_n-d_m} (1-p_2)^{n-m-d_n+d_m-1}\, dp_2\}.$$

$$(1-\theta_2)^{n-m-d_n+d_m-1} \theta_2^{S_n-S_m-1}\, d\theta_2$$

$$= k_3 \sum_{m=1}^{n-1} I_4(m)\, I_5(m)$$

Also, $h_4(\underline{X}) = \sum_{m=1}^{n-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 L_0(\theta_1, p_1, m|\underline{X}) g_{01}(\theta_1, p_1, m)\, d\theta_1\, dp_1$

$$= k_3 \sum_{m=1}^{n-1} \int_0^1 \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}-1} \, \mathrm{d}p_1\}$$

$$(1-\theta_1)^{m-\mathrm{d_m}-1} {\theta_1}^{\mathrm{S_m}-1} \, \mathrm{d}\theta_1$$

$$= k_3 \sum_{m=1}^{n-1} I_6(\mathrm{m})$$

For the integral I_4 (m),

$$\mathrm{I}_4(\mathrm{m}) = \int_0^1 \{\int_0^1 (p_2 - \theta_2(1-p_2))^{\mathrm{d_n}-\mathrm{d_m}} (1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}-1} \, \mathrm{d}p_2\}$$

$$(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}-1} {\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1} \, \mathrm{d}\theta_2$$

$$= \int_0^1 (1-\theta_2)^{n-m-1} {\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1}$$

$$\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m})\, {}_2F_1[1, \mathrm{d_m}-\mathrm{d_n}, 1+\mathrm{d_m}-\mathrm{d_n}-m+n, \tfrac{\theta_2}{\theta_2-1}] \, \mathrm{d}\theta_2$$

$$= \Gamma(n-m-\mathrm{d_n}+\mathrm{d_m}) \int_0^1 (1-\theta_2)^{n-m-1} {\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1}$$

$${}_2F_1[1, \mathrm{d_m}-\mathrm{d_n}, 1+\mathrm{d_m}-\mathrm{d_n}-m+n, \tfrac{\theta_2}{\theta_2-1}] \, \mathrm{d}\theta_2 \qquad (24)$$

where ${}_2F_1[1, \mathrm{d_m}-\mathrm{d_n}, 1+\mathrm{d_m}-\mathrm{d_n}-m+n, \tfrac{\theta_2}{\theta_2-1}]$ is a hypergeometric function as explained earlier.

For the integral $I_5(\mathrm{m})$,

$$I_5(\mathrm{m}) = \int_0^1 \{\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}-1} \, \mathrm{d}p_1\} (1-\theta_1)^{m-\mathrm{d_m}-1} {\theta_1}^{\mathrm{S_m}-1} \, \mathrm{d}\theta_1$$

$$= \int_0^1 \{\int_0^1 (p_1 - \theta_1(1-p_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}-1} \, \mathrm{d}p_1\} (1-\theta_1)^{m-\mathrm{d_m}-1} {\theta_1}^{\mathrm{S_m}-1} \, \mathrm{d}\theta_1$$

$$= \int_0^1 (1-\theta_1)^{m-1} {\theta_1}^{\mathrm{S_m}-1} \Gamma(m-\mathrm{d_m})$$

$${}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}]\, \mathrm{d}\theta_1$$

$$= \Gamma(m-\mathrm{d_m}) \int_0^1 (1-\theta_1)^{m-1} {\theta_1}^{\mathrm{S_m}-1} \Gamma(m-\mathrm{d_m})$$

$${}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}]\, \mathrm{d}\theta_1 \qquad (25)$$

where ${}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}]$ is a hyper geometric function as explained earlier.

For the integral $I_6(\mathrm{m})$, we have,

$$I_6(\mathrm{m}) = \int_0^1 \{\int_0^1 (p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}-1}\, \mathrm{d}p_1\} (1-\theta_1)^{m-\mathrm{d_m}-1} {\theta_1}^{\mathrm{S_m}-1}\, \mathrm{d}\theta_1$$

$$= \int_0^1 \{\int_0^1 (p_1-\theta_1(1-p_1))^{\mathrm{d_m}} (1-p_1)^{m-\mathrm{d_m}-1}\, \mathrm{d}p_1\} (1-\theta_1)^{m-\mathrm{d_m}-1} {\theta_1}^{\mathrm{S_m}-1}\, \mathrm{d}\theta_1$$

$$= \int_0^1 (1-\theta_1)^{m-1} {\theta_1}^{\mathrm{S_m}-1} \Gamma(m-\mathrm{d_m})$$

$${}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}]\, \mathrm{d}\theta_1$$

$$= \Gamma(m-\mathrm{d_m}) \int_0^1 (1-\theta_1)^{m-1} {\theta_1}^{\mathrm{S_m}-1} \Gamma(m-\mathrm{d_m})$$

$${}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}]\, \mathrm{d}\theta_1 \qquad (26)$$

where ${}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}]$ is a hyper geometric function as explained earlier.

8. MARGINAL POSTERIOR DENSITY OF 'm' UNDER NON-INFORMATIVE PRIOR AND UNDER NULL HYPOTHESIS

Now, we shall derive the posterior density of a change point 'm' of the model under non-informative prior and null hypothesis $\boldsymbol{H_0}$.

Now the joint posterior density of θ_1 *and* θ_2 is obtained using the joint posterior density of $\theta_1, \theta_2, p_1, p_2$ and m under non-informative prior given earlier and integrating out p_1 and p_2, we get,

$$g_2(\theta_1, \mathrm{m}|\underline{\mathrm{X}}) = \int_0^1 g_2(\theta_1, p_1, m|\underline{\mathrm{X}})\, \mathrm{d}p_1$$

$$= (1-\theta_1)^{m-\mathrm{d_m}-1}\theta_1^{\mathrm{S_m}-1}$$

$$\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}}(1-p_1)^{m-\mathrm{d_m}-1}\, \mathrm{d}p_1 \;\; h_5^{-1}(\underline{\mathrm{X}})$$

$$= \Gamma(m-\mathrm{d_m})(1-\theta_1)^{m-1}\theta_1^{\mathrm{S_m}-1}$$

$$ {}_2F_1\left[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \frac{\theta_1}{\theta_1-1}\right] h_5^{-1}(\underline{\mathrm{X}})$$

$$= k_2(1-\theta_1)^{m-1}\theta_1^{\mathrm{S_m}-1}$$

$$ {}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}] \;\; h_5^{-1}(\underline{\mathrm{X}}) \qquad (27)$$

where $k_2 = \Gamma(m-\mathrm{d_m})$ (28)

The marginal posterior density of θ_1 is obtained using joint posterior density of θ_1 *and* θ_2.

$$g_2(\theta_1, \mathrm{m}|\underline{\mathrm{X}}) = \sum_{m=1}^{n-1} g_2\left(\theta_1, \mathrm{m}\middle|\underline{\mathrm{X}}\right)$$

$$= k_2(1-\theta_1)^{m-1}\theta_1^{\mathrm{S_m}-1}$$

$$ {}_2F_1[1, -\mathrm{d_m}, 1-\mathrm{d_m}+m, \tfrac{\theta_1}{\theta_1-1}] \;\; h_5^{-1}(\underline{\mathrm{X}}) \qquad (29)$$

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The marginal posterior density of m is obtained using the joint posterior density of θ_1 , $and\, m$ and integrating with respect to θ_1 leads to the marginal posterior density of change point m. So, we have $g_2(m\,|\underline{X}\,) = \; I_3(\mathrm{m})\, I_4(\mathrm{m})\, {h_5}^{-1}(\underline{\mathrm{X}})$ (30)

9. MARGINAL POSTERIOR DENSITY OF 'm' UNDER NON-INFORMATIVE PRIOR AND UNDER ALTERNATIVE HYPOTHESIS

Finally, we shall derive the posterior density of a change point 'm' of the model under non-informative prior and alternative hypothesis $\boldsymbol{H_1}$.

The joint prior density of parameters using non-informative prior is as under:

$$g_{21}(\theta_1,\theta_2,p_1,p_2,m) = \frac{1}{(1-\theta_1)\theta_1(1-\theta_2)\theta_2(1-p_1)(1-p_2)} \qquad (31)$$

The joint posterior density of parameters $\theta_1,\theta_2,p_1,p_2$ and m under non-informative prior is obtained using the likelihood function and the joint prior density of the parameters under non-informative prior as under:

$$g_2\big(\theta_1,\theta_2,p_1,p_2,m\big|\underline{\mathrm{X}}\big) = L_1\big(\theta_1,\theta_2,p_1,p_2,m\big|\underline{\mathrm{X}}\big)\; g_{21}(\theta_1,\theta_2,p_1,p_2,m)\; {h_5}^{-1}(\underline{\mathrm{X}})$$

Now the joint posterior density of $\theta_1,\theta_2\; and\; m$ is obtained using the joint posterior density of $\theta_1,\theta_2,p_1,p_2$ and m under non-informative prior and integrating out p_1 and p_2 and then taking summation over 'm', we get,

$$g_2(\theta_1,\theta_2\;\mathrm{m}|\underline{\mathrm{X}}) = \int_0^1\int_0^1 g_2(\theta_1,\theta_2,p_1,p_2,m|\underline{\mathrm{X}})\,\mathrm{d}p_1\,\mathrm{d}p_2$$

$$= (1-\theta_1)^{m-\mathrm{d_m}-1}{\theta_1}^{\mathrm{S_m}-1}(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}-1}{\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1}$$

$$\int_0^1 (p_1 + (1-p_1)(1-\theta_1))^{\mathrm{d_m}}(1-p_1)^{m-\mathrm{d_m}-1}\,\mathrm{d}p_1$$

$$\int_0^1 (p_2 + (1-p_2)(1-\theta_2))^{\mathrm{d_n}-\mathrm{d_m}} (1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}-1}\, \mathrm{d}p_2 \;\; {h_5}^{-1}(\underline{\mathrm{X}})$$

$$= \Gamma(m-\mathrm{d_m})\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m})(1-\theta_1)^{m-1}{\theta_1}^{\mathrm{S_m}-1}$$

$${}_2F_1[1,-\mathrm{d_m},1-\mathrm{d_m}+m,\tfrac{\theta_1}{\theta_1-1}](1-\theta_2)^{n-m-1}{\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1}$$

$${}_2F_1\left[1,\mathrm{d_m}-\mathrm{d_n},1+\mathrm{d_m}-\mathrm{d_n}-m+n,\tfrac{\theta_2}{\theta_2-1}\right] {h_5}^{-1}(\underline{\mathrm{X}})$$

$$= k_4(1-\theta_1)^{m-1}{\theta_1}^{\mathrm{S_m}-1}$$

$${}_2F_1[1,-\mathrm{d_m},1-\mathrm{d_m}+m,\tfrac{\theta_1}{\theta_1-1}](1-\theta_2)^{n-m-1}{\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1}$$

$${}_2F_1\left[1,\mathrm{d_m}-\mathrm{d_n},1+\mathrm{d_m}-\mathrm{d_n}-m+n,\tfrac{\theta_2}{\theta_2-1}\right] {h_5}^{-1}(\underline{\mathrm{X}}) \qquad (32)$$

where, $k_4 = \Gamma(m-\mathrm{d_m})\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m})$ (33)

The joint posterior density of θ_1 is obtained using joint posterior density of θ_1, θ_2 *and* m and then integrating out θ_2, we get,

$$g_2(\theta_{1,}\mathrm{m}|\underline{\mathrm{X}}) = \sum_{m=1}^{n-1} \int_0^1 g_2\left(\theta_1,\theta_2\middle|\underline{\mathrm{X}}\right) \mathrm{d}\theta_2$$

$$= \sum_{m=1}^{n-1} k_4(1-\theta_1)^{m-1}{\theta_1}^{\mathrm{S_m}-1}$$

$${}_2F_1[1,-\mathrm{d_m},1-\mathrm{d_m}+m,\tfrac{\theta_1}{\theta_1-1}]$$

$$\int_0^1 (1-\theta_2)^{n-m-1}{\theta_2}^{\mathrm{S_n}-\mathrm{S_m}-1}(-\theta_2)^{\mathrm{d_n}-\mathrm{d_m}}$$

$${}_2F_1\left[1,\mathrm{d_m}-\mathrm{d_n},1+\mathrm{d_m}-\mathrm{d_n}-m+n,\tfrac{\theta_2}{\theta_2-1}\right] \mathrm{d}\theta_2 \;\; {h_5}^{-1}(\underline{\mathrm{X}}) \qquad (34)$$

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The marginal posterior density of 'm' is obtained using the joint posterior density of θ_1 *and* m and integrating with respect to θ_1 leads to the marginal posterior density of change point 'm' under alternative hypothesis , say, $g_{21}(m|\underline{x})$, which is given by

$$g_{21}(m\,|\underline{X}\,) = I_7(\mathrm{m})I_8(\mathrm{m})\,h_5{}^{-1}(\underline{\mathrm{X}})$$

where,

$$I_7(\mathrm{m}) = \int_0^1\{\int_0^1(p_1+(1-p_1)(1-\theta_1))^{\mathrm{d_m}}(1-p_1)^{m-\mathrm{d_m}-1}\,\mathrm{d}p_1\}(1-\theta_1)^{m-\mathrm{d_m}-1}\theta_1{}^{\mathrm{S_m}-1}\,\mathrm{d}\theta_1$$

$$= \int_0^1\{\int_0^1(p_1-\theta_1(1-p_1))^{\mathrm{d_m}}(1-p_1)^{m-\mathrm{d_m}-1}\,\mathrm{d}p_1\}(1-\theta_1)^{m-\mathrm{d_m}-1}\theta_1{}^{\mathrm{S_m}-1}\,\mathrm{d}\theta_1$$

$$= \int_0^1(1-\theta_1)^{m-1}\theta_1{}^{\mathrm{S_m}-1}\Gamma(m-\mathrm{d_m})$$

$$_2F_1[1,-\mathrm{d_m},1-\mathrm{d_m}+m,\tfrac{\theta_1}{\theta_1-1}]\,\mathrm{d}\theta_1$$

$$= \Gamma(m-\mathrm{d_m})\int_0^1(1-\theta_1)^{m-1}\theta_1{}^{\mathrm{S_m}-1}\Gamma(m-\mathrm{d_m})$$

$$_2F_1[1,-\mathrm{d_m},1-\mathrm{d_m}+m,\tfrac{\theta_1}{\theta_1-1}]\,\mathrm{d}\theta_1 \qquad (35)$$

where $_2F_1[1,-\mathrm{d_m},1-\mathrm{d_m}+m,\frac{\theta_1}{\theta_1-1}]$ is a hyper geometric function as explained earlier.

For the integral $I_8(m)$,

$$\mathrm{I_8(m)} = \int_0^1\{\int_0^1(p_2-\theta_2(1-p_2))^{\mathrm{d_n}-\mathrm{d_m}}(1-p_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}-1}\,\mathrm{d}p_2\}$$

$$(1-\theta_2)^{n-m-\mathrm{d_n}+\mathrm{d_m}-1}\theta_2{}^{\mathrm{S_n}-\mathrm{S_m}-1}\,\mathrm{d}\theta_2$$

$$= \int_0^1(1-\theta_2)^{n-m-1}\theta_2{}^{\mathrm{S_n}-\mathrm{S_m}-1}$$

$\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m})\,{}_2F_1[1,\mathrm{d_m}-\mathrm{d_n},1+\mathrm{d_m}-\mathrm{d_n}-m+n,\frac{\theta_2}{\theta_2-1}]\,\mathrm{d}\theta_2$

$=\Gamma(n-m-\mathrm{d_n}+\mathrm{d_m})\int_0^1(1-\theta_2)^{n-m-1}\theta_2^{\,\mathrm{S_n-S_m}-1}$

$${}_2F_1[1,\mathrm{d_m}-\mathrm{d_n},1+\mathrm{d_m}-\mathrm{d_n}-m+n,\frac{\theta_2}{\theta_2-1}]\,\mathrm{d}\theta_2 \quad (36)$$

where ${}_2F_1[1,\mathrm{d_m}-\mathrm{d_n},1+\mathrm{d_m}-\mathrm{d_n}-m+n,\frac{\theta_2}{\theta_2-1}]$ is a hyper geometric function as explained earlier.

10. NUMERICAL STUDY

Her, we have generated 20 random observations from the proposed change point model. The first 10 observations from Zero Inflated Geometric Distribution where parameter values are $\theta_1=0.4$ and p_1 = 0.5. The next 10 observations are from the same distribution with parameter values $\theta_2=0.5$ and p_2 = 0.6. Here, we note that p_1 *and* p_2 are the random observations from the Beta Distribution with mean $\mu_1=0.4$ *and* $\mu_2=0.9$ and the value of standard deviation is $\sigma=$ 0.1 respectively. The observations are given below in Table 1.

TABLE 1

Generated Observations from the proposed change point model of Zero Inflated Geometric Distribution

I	1	2	3	4	5	6	7	8	9	10
X_i	0	1	1	0	2	1	0	3	1	3
I	11	12	13	14	15	16	17	18	19	20
X_i	1	1	2	0	7	0	2	1	2	1

TABLE 2

Values of Bayes Estimates of Proportions

Prior	Bayes Estimates of Proportions	
	p_1	p_2
Informative	0.66	0.89
Non-Informative	0.57	0.85

TABLE 3

Frequency Distributions of Bayes Estimates of Proportions

Bayes Estimates	% Frequency for			
	0.1-0.3	0.3-0.5	0.5-0.7	0.7-0.9
$\hat{p}_{1E}$	14	06	74	05
$\hat{p}_{2E}$	12	02	06	83

11. CONCLUSIONS

The results shown in Table 2 indicate the values of Bayes Estimates of proportions under the informative and non-informative priors respectively and the values of frequency distribution of Bayes Estimates of proportions are shown below in Table 3. Finally, we conclude that performance of posterior means has better performance than those of the change point. 68% values of posterior mean are close to actual value of change point with correct choice of prior. 66% values of posterior median are close to actual value of change point with correct choice of prior. 67% values are close to actual value of change point with correct choice of prior. 69% values are close to actual value of change point with correct choice of prior.

12. REFERENCES

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