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Existence of Solution of Optimal Control Problem Under Fractional Integro Differential Equations

Mathematics

KEYWORDS : Fractional calculus, Caputo fractional Derivative, Partial differential Equations, Optimal control, Existence

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ABSTRACT

In this article we deal with the existence of the solution of optimal control problem under the fractional Integro-partial differential equations with state function. Further, the Existence of the optimal control and necessary optimality conditions are proved.

INTRODUCTION

Fractional differential equations have become an engineering, Biophysics blood bank, biology, Control theory, Aero dynamics, Electron-analytical chemistry etc. The theory of differential equations of fractional order has recently received a lot of attention and no important objective of investigate in the fields of Physics, constitutes an important branch of non-linear analysis.

The most intriguing and useful applications of fractional derivatives and integrals in engineering and science have been found in the past one hundred years. In some cases, the mathematical notations evolved in order to better meet the requirement Caputo fractional derivative is of the best example and the most popular fractional operator among engineers and applied mathematicians.

Riemann-Liouville and Grunwald-Letnikov are the most popular mathematicians of using the definitions, notations of Fractional Calculus, Fractional derivatives. Particularly in the last decade of 20th century, numerous applications and physical manifestations of fractional calculus have been found. The fractional calculus by considering fractional derivatives into the variation integrals to be extremised. This occurs naturally in many problems of physics and mechanics.

Motivated by the previous literature, the purpose of this paper is to studying the existence of the control ω which maximize the cost functional,

$$F(w) = \int_0^{\infty} \left[G(w) + C w_x^2 (1, 0 \right] d\theta$$

where C is constant.

2. PRELIMINARIES

In this section, we give some definition and Lemmas which are used in this paper.

Definition 2.1

Let f : R+ → R be a continuous on R+. Then the Riemann – Lioville fractional of derivative order α is given by the expression

$$D_\theta^\alpha f(\theta) = \frac{1}{(n-\alpha)} \frac{d^n}{d\theta^n} \int_0^{\infty} (\theta - s)^{n-\alpha-1} f(s) ds \ \ \theta > 0 \qquad \text{... (1)}$$

Where α ∈ (n-1, n), n ∈ C.

2.2 Definition

Let f : R+ → R, be a continuous on R+ the Riemann-Liouville integral of order α is defined by

$$I_\theta^\alpha f(\theta) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (\theta - s)^{\alpha-1} f(s) ds.... \qquad (2)$$

2.3 Definition

Let f : R+ → R, then the expression be a continuous on R+ the Riemann-Liouville integral of order α is defined by

$$D_\theta^\alpha f(\theta) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\theta^n} \int_0^{\infty} (\theta - s)^{n-\alpha-1} f(s)^{(n)} ds \ \ \theta > 0$$

where α ∈ (n-1, n), n ∈ C, called the Caputo fractional Left derivative of α of f.

3. PROBLEM FORMULATION

We consider the linear fractional integro-differential equation,

$$\omega_\alpha(x,\theta) = \frac{\partial^\alpha w}{\partial\alpha}(x,\theta) = C^{\omega_{xx}}(x,\theta) + C(x)\omega(x,\theta)$$
$$+(p(x) + \int_o^x P(.)\,dW)\ \omega(0,\theta) + \int_0^x f(x,y)\ \omega(y,\theta)dy \qquad (3)$$

$$wx\ (0,\theta) = q\ w\ (0,\ \theta) \qquad (4)$$

$$wx\ (1,\theta) = v\ (t) \qquad (5)$$

for $x \in (0,1)$, $\theta > 0$, $C > 0$, $q \in R$.

where ν (t) is the control Input. The control objective is to stabilize the equilibrium w (x, δ) = 0 and W is the Brownian motion.

Theorem 3.1 The Transformation

$$L(x,\theta) = w(x,\vartheta) - \int_0^x r(x,y)w(y,\sigma)dy \qquad (6)$$

Transforms the system 3,4,5 into the system

$$\frac{\partial^\alpha L}{\partial\theta^\alpha} = cL_{xx}(x,\theta) - \beta L(x,\theta),\ x \in (0,1) \qquad (7)$$

$$Lx\ (0,\theta) = qL(0,\theta) \qquad (8)$$

$$L\ (1,\theta) = 0 \text{ or } Lx\ (1,\theta) = 0 \qquad (9)$$

Which is exponentially stable for β > cl2 where l = max [0, -q] and the transformer r(x,y) satisfies the hyperbolic partial differential equation.

$$C\frac{\partial^2 r}{\partial x^2} - C\frac{\partial^2 r}{\partial y}$$
$$= \left[h(y)+\beta\right]r(x,y) - f(x,y) + \int_y^x r(x,\mu)f(\mu,y)d\mu \qquad (9.1)$$

With the boundary conditions,

$$C\frac{\partial r}{\partial y}(x,0) = q\ \ Cr(x,0) + g(x) + \rho W(x) - \int_0^x g(y) + \rho W(y)r(x,y)dy \qquad (10)$$

$$r(x,x) = -\frac{1}{2c}\int_0^x \left[h(y)+\beta\right]dy \qquad (11)$$

Proof

Differentiating (6) w. r.t 'θ' we get

$$\frac{\partial^\alpha L(x,\theta)}{\partial\theta^\alpha} = \frac{\theta^\alpha w(x,\theta)}{\partial\theta^\alpha} -$$

$$\frac{\partial^\alpha L(x,\theta)}{\partial\theta^\alpha} = \frac{\partial^\alpha w(x,\theta)}{\partial\theta^\alpha} - \int_0^x r(x,y)\left[C\frac{\partial^2 u}{\partial y^2}(y,\theta) + h(y)w(y,\theta)\right.$$

$$\left. + \left\{g(y) + pW(y)\right\}w(0,\theta) + \int_0^y f(y,\mu)w(\mu,\theta)\right]dy \qquad (12)$$

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial^2 w(x,\theta)}{\partial x^2} - r(x,x)\frac{\partial w}{\partial x}(x,\theta) - w(x,\theta)r(x,x)\frac{\partial r}{\partial x}.w(x,0)$$
$$-\int_0^y \frac{\partial^2 r(x,y)}{\partial x^2}w(y,\vartheta)dy$$

(13)

Where

$$\frac{dr(x,x)}{dx} = \frac{\partial r(x,x)}{\partial x} + \frac{\partial r(x,x)}{\partial y}$$

$$\frac{dr(x,x)}{\partial x} = \frac{\partial r(x,y)}{\partial x}\ ;\ \frac{\partial r(x,x)}{\partial y} = \frac{\partial r(x,x)}{\partial y}$$

$$y \to x' \qquad y \to x'$$

Substituting 12 & 13 is 7, 8, 9 and 3, 4, 5, we get the following equation,

$$\int_0^x\left[f(x,y) - c\frac{\partial^2 r}{\partial y^2}(y,\theta) - h(y)r(x,y) - \beta r(x,y)\right.$$
$$\left. + c\frac{\partial^2 r}{\partial x^2}(x,y) - \int_y^x r(x,\mu)f(\mu,y)d\mu\right]w(y,\theta)$$

$$+\left[h(x) + c\frac{\partial r}{\partial y}(x,x) + c\frac{\partial r}{\partial x}(x,y) + c\frac{\partial r}{\partial x}(x,x) + \beta\right]\omega(y,\theta)$$

$$p(x) + \rho W(x) + q\ cr(x,0) - c\frac{\partial r}{\partial y}(x,\theta) - \int_0^x\left\{p(y) + pW(y)\right\}r(x,)dy\Bigg]w(0,\theta)$$

For the equation to be verified = 0 for all w (x, θ) the system 10, 12, 12 must be satisfied.

Hence the Theorem.

4. Analysis of Inverse Optimal Control

In this section, we discuss the solution procedure of Inverse optimal control problem. We design the controller to stabilization of the system of equation.

$$\frac{\partial^\alpha w(x,\theta)}{\partial\theta^\alpha} = a\frac{\partial^\alpha u(x,\theta)}{\partial x^2} + h(x)w(x,\theta)$$
$$+\left[p(x) + \int_0^x P(.)dW\right]w(0,\theta) + \int_0^x f(x,y)w(y,\theta)dy \qquad (4.1)$$

$$\frac{\partial w}{\partial x}(x,\theta) = q\ \text{w}\ (0,\theta) \qquad (4.2)$$

$$w(1,\theta) = \nu(\theta),\ \text{or}\ \frac{\partial w}{\partial x}(1,\theta) = \nu(t) \qquad (4.3)$$

Further, the controller also minimizes some meaningful cost functional.

Theorem 4.1

Consider the system 4.1, 4.2, 4.3 with the associated functional

$$f(\omega) = \int_0^\infty G(\omega) + C\omega_x^2(1,\theta)d\ \theta$$

$$f(w) = \int_0^\infty G(w) + C\left(\frac{\partial^2 w}{\partial x^2}\right)^2 d\ \theta \qquad (4.4)$$

Where,

$$G(w) = \frac{1}{C}\delta^2 L^2(1,\theta) + 2\delta\left[L(1,\theta)\int_0^1 l_x(1,y)L(y,\theta)dy + \delta L^2(0,\theta)\right.$$

$$\left. + l(1,1)L^2(1,\theta) + \frac{\beta}{c}\int_0^1 L^2(x,\theta)dx + \int_0^1\left(\frac{\partial L}{\partial x}\right)^2 dx\right] \qquad (4.5)$$

$$G(w) \geq \frac{1}{C}\delta(\delta-2)L^2(1,t) + \frac{2\delta}{c}\{\beta - c(2\bar{q}^2+1)\}\int_0^1 L^2(x,\theta)dx + \delta\int_0^1\left(\frac{\partial L}{\partial x}\right)^2 dx \qquad (4.6)$$

and

$$C = \left[l(1,1)\int_0^1\left(\frac{\partial l(1,y)}{\partial x}\right)^2 dy + \bar{q}\right]^{-1} > 0$$

For

$$\text{for } \bar{q} = \max(0,-q),\ \beta > C(2\bar{q}^2+1) \text{ and } \delta \geq 2$$

Then the control,

$$\frac{\partial \bar{\omega}}{\partial x}(1,\theta) = -\frac{\delta}{C}\left(w(1,\theta) - \int_0^1 r(1,y)w(y,\theta)dy\right) \qquad (4.8)$$

Minimizes the cost functional

$$f(w) = \int_0^\infty G(w) + C\left(\frac{\partial w}{\partial \theta}(1,\theta)\right)^2 d\theta$$

Proof

Let the function A(t) defined by

$$A(t) = \int_0^1 L(x,\theta)\frac{\partial^\alpha L}{\partial\theta^\alpha}(x,\theta)dx \qquad (4.9)$$

(ie) using

$$\frac{\partial^\alpha L}{\partial\theta^\alpha} = c\frac{\partial^2 L}{\partial x^2}(x,\theta) - \beta L(x,\theta);\ x \in (0,1)$$

and $\omega(x,0) = L(x,\theta) + \int_0^x l(x,y)L(y,0)dy$ in above we get,

$$A(t) = c\int_0^1 L(x,\theta)\frac{\partial^2 L(x,\theta)}{\partial x^2}dx - \beta\int_0^1 L^2(x,\theta)dx$$

$$= cL(1,\theta)\frac{\partial\omega}{\partial x}(x,\theta) - \beta L(1,\theta)\int_0^1 \frac{\partial l}{\partial x}(1,y)L(y,\theta)dy$$

$$= cl(1,1)L^2(1,\theta) - c\bar{q}^2(0,\theta) - c\int_0^1\left(\frac{\partial V}{\partial x}\right)dx(x,\theta) - \beta\int_0^1 L^2(x,\theta)dx \qquad (4.10)$$

Now, equation (4.5) can be written as,

$$G(\omega) = \frac{1}{C}\delta^2 L^2(1,\theta) + 2\delta\left[L(1,\theta)\frac{\partial\omega}{\partial x}(1,\theta) - \frac{1}{c}A(t)\right]$$

Substituting the above equation (4.9) in to the cost functional

$$f(\omega) = \int_0^\infty \left[\frac{\delta^2}{c}L^2(1,\theta) + 2\beta\left(L(1,\theta)\frac{\partial\omega(1,\theta)}{\partial x} - \frac{A(t)}{c}\right) + C\left(\frac{\partial w}{\partial x}(1,\theta)\right)^2\right]d\theta$$

$$= \frac{2\delta}{c}\left\{A(0) - A(\infty)\right\} + C\int_0^\infty\left\{\frac{\partial w}{\partial x}(1,\theta) + \frac{\delta}{C}(1,\theta)\right\}^2 d\theta$$

$$= \frac{2\delta}{c}\left\{A(0) - A(\infty)\right\} + C\int_0^\infty\left\{\frac{\partial w}{\partial x}(1,\theta) - \frac{\overline{\delta w}}{C}(1,\theta)\right\}^2 d\theta \qquad (12)$$

Using (4.5) and (4.7), the Cauchy-Shcartz inequality and Agmon's inequality, we get

$$\underset{x\in[o,1]}{Max} L^2(x,\theta) \leq L^2(1,\theta) + 2\left[\int_0^1 L^2(x,\theta)dx\right]^{1/2}\left(\int_0^1 L^2 x(x,\theta)dx\right)^{1/2} \qquad (4.13)$$

(ie) the equation (4.6) proved.

So G is a +ve definite functional which makes

$$f(w) = \int_0^\infty G(w) + Cw_x^2(1,\theta), d\theta$$

A reasonable cost which puts penalty on both state and the control form of the equation (4.11), Now, We have,

$$A(t) = \frac{\delta c}{2C}L^2(1,\theta) + cL(1,\theta)\frac{\delta w}{\partial x}(1,\theta) - \frac{c}{2\delta}G$$

$$\leq \left\{\beta - c(2\bar{q}^2+1)\right\}\int_0^1 L^2(x,\theta)dx - \frac{c}{2}\int_0^1 L_x^2(x,\theta)dx$$

Which shows that the controller $\bar{\omega}_x(1,\theta)$ *satisfies the system* 7, 8, 9 *and thus the origina*

$$\frac{\delta^\alpha w(x,\theta)}{\partial\theta^\alpha} = C\frac{\delta^2 w}{\partial x^2}(x,\theta) + h(x), w(x,\theta)$$

$$+\left(p(x) + \int_o^x P(.)dW\right)\omega(0,\theta) + \int_0^x f(x,y)\omega(y,\theta)dy \qquad (4.14)$$

$$\omega x(0,\theta) = q\, w(0,\theta)$$

(4.15)

and $\omega(1,\theta) = V(t) \Rightarrow wx(1,\theta) = v(t)$ (4.16)

Now putting $A(\infty) = 0$ in equation (4.13) which completes the proof.

5. CONCLUSION

In this article, a solution of an Inverse optimal control problem has been proved using with the state function governed by Fractional Partial Differential Equation. Moreover, the existence result of solution based on hyperbolic differential equation method has been established.

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