A Cost Function for Complex Systems
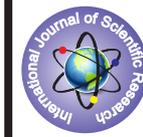
Prashanth Alluvada | Electrical and Computer Engineering Jimma University Jimma, Ethiopia

ABSTRACT We introduce a new cost function over graphs built through the graph's characteristic path length (CPL) and clustering coefficient (CC). The function describes the extreme graph configurations ("boundaries") corresponding to the specified property values (CC, CPL) so that several dynamics of the CC and CPL based graphs occur within the bounding graphs described by the cost function. For graphs that are rewired with probability p, we show through computation that as the rewiring probability is varied, the cost function passes a global minimum point.

1. Introduction

Dynamical systems of varied types occur in the universe. Complex systems are large interconnected (dynamical) systems consisting of several interacting units. Because of large size, such systems possess extra degrees of freedom through which new dynamics can occur. Complex systems are used to analyze cortical events responsible for cognition, emergent organization, self-organized criticality among others. The main focus of this article however, is the underlying graph through which complex systems are described.

In graphs that occur in the real world, natural rewiring occurs through principles derived from their ambience of occurrence. Graph rewiring occurs in social networks, computer networks, human brain (learning process among others). A rewiring principle conducts the rewiring and the rewiring occurs for until the graph reaches a steady state or a state of inertia. The graph is dormant with no rewiring once the state of inertia is reached for until the ambient conditions do not change. The rewiring principle itself is derived from the ambience in which the graph occurs. When a change occurs in the ambient conditions, rewiring restarts and proceeds for until a new steady state is reached. Social networks adapt over time to minimize the path length and increase the average clustering, connectivity of the computers over the internet is a scale-free distribution, in the prenatal cortex, it is proposed that a dynamic synchronization among chaotically firing neurons causes small world connectivity to develop through the neural network.
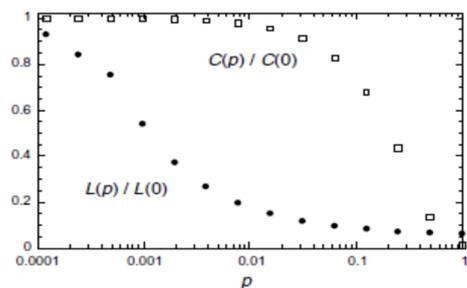
In these (rewiring) dynamics it is unclear if the steady state reached is the best possible state a graph can admit under the given conditions or how close to the best configuration the steady states can reach to at the end of a rewiring procedure. We make precise the "best configuration" in this article. The best configuration is an intrinsic property of the graph of a given size and connection density.

2. Graphs, Rewiring and Small Worlds

It is proposed that the small world graphs are intermediate between that of regular graphs and random graphs - they are highly clustered like the regular graph and possess a short CPL like the random graph ([1]). Rewiring with probability p has introduced controlled amounts of order ("information" through rewiring) into a regular graph and pushed it closer to a small world graph. It is possible through a similar rewiring procedure to introduce controlled amounts of disorder and push the graph farthest from the small world. If this graph is a large world, by analogy with small world, then properties dual to those of the small world graphs can occur over the large world graphs. Such properties can lead to symmetry relationships among graph subsets. Important properties such as human brain connectivity, small world networks, scale-free distributions all occur within a class of graphs whose boundary is described by a cost function.

The rewiring algorithm described in [1] is listed here for completeness: algorithm describes a random rewiring procedure for interpolating a regular ring lattice and a random network without altering the number of vertices or edges in the graph. Begin with a ring of n vertices each connected to its k nearest neighbors by

undirected edges. Choose a vertex and the edge that connects it to its nearest neighbor in a clockwise sense. With probability p reconnect this edge to a vertex chosen uniformly at random over the entire ring with duplicate edges forbidden; otherwise leave the edge in place. Repeat this process by moving clockwise over the ring considering each vertex in turn until one lap is completed. Next consider the edges that connect vertices to their second nearest neighbors clockwise. As before randomly rewire each of these edges with probability p and continue the process, circulating around the ring and proceeding outward to the more distant neighbors after each lap until each edge in the original lattice has been considered once. (As there are nk/2 edges in the entire graph, the rewiring process stops after k/2 laps.)

Figure Characteristic path length $L(p)$ and clustering coefficient $C(p)$ for the family of randomly rewired graphs described in Fig. 1. Here L is defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices. The clustering coefficient $C(p)$ is defined as follows. Suppose that a vertex v has k_v neighbours; then at most $k_v(k_v - 1)/2$ edges can exist between them (this occurs when every neighbour of v is connected to every other neighbour of v). Let C_v denote the fraction of these allowable edges that actually exist. Define C as the average of C_v over all v. For friendship networks, these statistics have intuitive meanings: L is the average number of friendships in the shortest chain connecting two people; C_v reflects the extent to which friends of v are also friends of each other; and thus C measures the cliquishness of a typical friendship circle. The data shown in the figure are averages over 20 random realizations of the rewiring process described in Fig. 1, and have been normalized by the values $L(0)$, $C(0)$ for a regular lattice. All the graphs have $n = 1,000$ vertices and an average degree of $k = 10$ edges per vertex. We note that a logarithmic horizontal scale has been used to resolve the rapid drop in $L(p)$, corresponding to the onset of the small-world phenomenon. During this drop, $C(p)$ remains almost constant at its value for the regular lattice, indicating that the transition to a small world is almost undetectable at the local level.

Fig.1. The CC and CPL of the ring shaped lattice, at the end of the rewiring, as a function of the rewiring probability p ([1]).

The rewiring algorithm described in [1] is listed here for completeness: algorithm describes a random rewiring procedure for interpolating a regular ring lattice and a random network without altering the number of vertices or edges in the graph. Begin with a ring of n vertices each connected to its k nearest neighbors by undirected edges. Choose a vertex and the edge that connects it to its nearest neighbor in a clockwise sense. With probability p reconnect this edge to a vertex chosen uniformly at random over the entire ring with duplicate edges forbidden; otherwise leave the edge in place. Repeat this process by moving clockwise over the ring considering each vertex in turn until one lap is completed. Next consider the edges that connect vertices to their second nearest neighbors

clockwise. As before randomly rewire each of these edges with probability p and continue the process, circulating around the ring and proceeding outward to the more distant neighbors after each lap until each edge in the original lattice has been considered once. (As there are nk/2 edges in the entire graph, the rewiring process stops after k/2 laps.)

3. The Cost Function

Given a connected graph of a size and connection density, the path length between two vertices is defined as the smallest number of edges connecting them. An average of the path lengths over the entire graph, taking every pair of the nodes, is defined as the graph's characteristic path length or CPL. We computed the path length using the Dijkstra algorithm. On the other hand, the clustering at a point (at a node) within the graph measures the extent to which neighbors of the node are neighbors among each other. When the clustering is averaged over the entire graph, we get the clustering coefficient (CC) for the graph. Using the definitions of CC and CPL on a graph, we define the following cost function

$$E = (\alpha\ CPL + \beta)^2 + (\gamma\ CC + \delta)^2 \qquad (3.1)$$

The $\alpha, \beta, \gamma, \delta$ are suitably chosen constants. The function may be minimized (optimized) using any of the random variable based search methods. The minimum (extremum) graph returned by the function is the smallest for the set values of the $\alpha, \beta, \gamma, \delta$ When the graph's (G(V,E)) connection density and node numbers are specified, then the connectivity returned by the cost function for a given $\alpha, \beta, \gamma, \delta$ is the best possible. However, the same graph when rewired at probability p may not reach the same CC and CPL values at the steady state of rewiring, as the graph returned by the cost function. On the other hand, when the cost function itself is evaluated over the CC and CPL's the same graph (i.e. G(V,E)) has reached through rewiring, and plotted as a function of the rewiring probability, then the cost function exhibits a global minimum point (Fig. 2)

$$\alpha = 1, \beta = 0, \gamma = -1, \delta = 1.5 \qquad (3.2)$$
$$E = CPL^2 + (1.5 - CC)^2 \qquad (3.3)$$

The cost function may be minimized by adding constraints of various types, such as constraint on the node-connectivity distribution requiring scale-freeness, or constraints requiring specific types of symmetry of the graph.
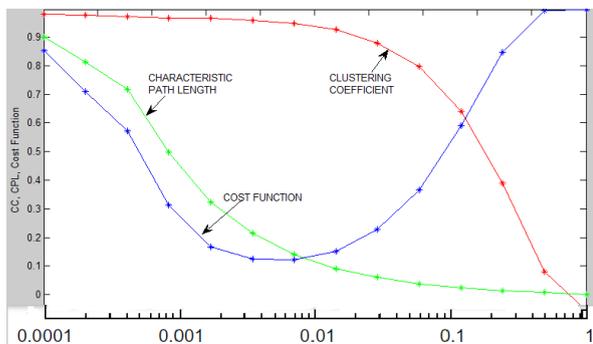
Fig. 2. The cost function (Eqn. 3.1) is evaluated over the CC and CPL obtained by rewiring the graph with probability p. The cost function exhibits a global minimum point. Abscissa has rewiring probability, the ordinate shows, CC (red), CPL (green) and cost function (blue).

4. Conclusion and Future Work

We expect that certain important graph properties may be captured through the cost function at the minimizing rewiring probability. In future articles we'll demonstrate some properties of the minimum graphs.

References

1. Watts, DJ and Strogatz, SH (1998), "Collective Dynamics of Small World Networks," Nature, Vol 393, June 1998.

2. *Author was formerly at RIKEN Brain Science Institute, Saitama, Japan. Author discovered the minimum point of the cost function in July 2003.