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BAYESIAN ANALYSIS OF EXPONENTIAL DISTRIBUTION BASED ON THE DOUBLE PRIOR SELECTION WITH DIFFERENT LOSS FUNCTIONS

Statistic

KEYWORDS: Prior distribution, posterior distribution, loss function, posterior risk.

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ABSTRACT Exponential distribution plays an important role in life testing problems. This paper provides prior selection for the unknown parameter of Exponential distribution using Gamma-Chi-square as double prior selection. The Bayes estimators under different loss function is compared.

1. INTRODUCTION:

Exponential distribution plays an important role in lifetime data analysis. Many authors have developed inference procedures for Exponential model. Abdul et al (2001) explored the relationship between and Bayesian and classical estimation. Bayes estimator of the standard exponential distribution with discussion on symmetric and asymmetric loss function was discussed by Ali et al(2005). Aslam et al (2010) described the relationship between Bayesian and classical estimation using uninformative priors for the parameter of the exponential model for time-to-failure data. Kazmi et al.(2012) compared class of life time distributions for Bayesian analysis. They studied properties of Bayes estimators of the parameter using different loss functions via simulated and real life data. Radha et al (2013) assumed generalized uniform-inverted gamma distribution as double priors for the parmater of Maxwell distribution to obtain Bayesian estimator.

In this paper, Gamma- Chi-square distribution is considered as double prior for single unknown parameter of the exponential distribution . Posterior distribution under this double prior is derived. Bayes estimator under the loss functions, squared error loss function and quadratic loss function is also derived. Comparison is done using posterior risk of these loss functions.

2. POSTERIOR DISTRIBUTION:

The posterior distribution for the unknown parameter of Exponential distribution under double prior namely Gamma-Chi-Square distribution is derived in the following section.

2.1. Likelihood function:

A random variable X is said to have an Exponential distribution with parameter θ and its probability density function is given by,

$$f(x,\theta) = \theta e^{-\theta x}, \qquad x > 0\ ,\ \theta > 0 \qquad (1)$$

The likelihood of the sample observations $\boldsymbol{x}$: $x_1, x_2 \dots, x_n$ is

$$L(\boldsymbol{x},\theta) = \prod_{i=1}^{n} f(x_i\ ,\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \qquad (2)$$

thus the maximum likelihood estimator for θ is $\hat{\theta} = \frac{1}{\bar{x}}$.

2.2 POSTERIOR DISTRIBUTION FOR GAMMA-CHI-SQUARE AS PRIOR:

The double prior distribution of θ is Gamma distribution with hyper parameters a_1 and b_1 is given by

$$f_{11}(\theta) = \frac{b_1^{a_1} \theta^{a_1 - 1} e^{-\theta b_1}}{\Gamma_{a_1}}\ ,\ \theta > 0, a_1 > 0, b_1 > 0 \qquad (3)$$

The second prior distribution of θ c_1; assumed as Chi-Square distribution with hyper parameter and its probability density function given by

$$f_{12}(\theta) = \frac{\theta^{\frac{c_1}{2} - 1} e^{-\frac{\theta}{2}}}{\Gamma(\frac{c_1}{2}) 2^{\frac{c_1}{2}}}\ , \qquad \theta > 0, c_1 > 0 \qquad (4)$$

Now we define the double prior for

$$f_1(\theta) \propto f_{11}(\theta)\ f_{12}(\theta)$$

$$\propto e^{-\theta(b_1 + \frac{1}{2})} \theta^{(a_1 + \frac{c_1}{2} - 1) - 1} \qquad (5)$$

The posterior distribution of x is

$$P(\theta/\boldsymbol{x}) \propto L(\boldsymbol{x},\theta)\ f_1(\theta)$$

$$\propto e^{-\theta(\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})}\ \theta^{(a_1 + n + \frac{c_1}{2} - 1) - 1}, \theta > 0$$

$$= k\ e^{-\theta(\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})}\ \theta^{(a_1 + n + \frac{c_1}{2} - 1) - 1} \qquad (6)$$

where $k^{-1} = \int_0^\infty e^{-\theta(\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})}\ \theta^{(a_1 + n + \frac{c_1}{2} - 1) - 1} d\theta$

on solving, $k = \frac{\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})^{(a_1 + n + \frac{c_1}{2} - 1)}}{\Gamma(a_1 + n + \frac{c_1}{2} - 1)}$ (7)

using (7) in (6), we have

$$P(\theta/\boldsymbol{x}) = \frac{\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})^{(a_1 + n + \frac{c_1}{2} - 1)}}{\Gamma(a_1 + n + \frac{c_1}{2} - 1)}\ e^{-\theta(\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})}\ \theta^{(a_1 + n + \frac{c_1}{2} - 1) - 1}$$

$$\propto e^{-\theta(\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})}\ \theta^{(a_1 + n + \frac{c_1}{2} - 1) - 1}, \theta > 0 \qquad (8)$$

$$= \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1}\ e^{-\theta\beta_1}\ \theta^{\alpha_1 - 1}, \theta > 0 \qquad (9)$$

which is the kernel density of the gamma distribution with the parameters $\alpha_1 = (a_1 + n + \frac{c_1}{2} - 1)$ and $\beta_1 = (\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})$. Hence, we find that the posterior distribution of θ is Gamma(α_1, β_1).

3. BAYESIAN ESTIMATION FOR DIFFERENT LOSS FUNCTIONS:

Let $L(\theta, \theta^*)$ be a loss function and $E(L(\theta, \theta^*))$ is a risk function . Bayes decision is a decision which minimizes risk function and gives best decision. If the decision is choice of an estimator then Bayes decision is a Bayes estimator.

3.1 Squared Error Loss Function (SELF):

The squared error loss function is defined as:

$$L(\theta, \theta_{SELF}) = (\theta - \theta_{SELF})^2 \qquad (10)$$

3.1.1 Bayes estimator under SELF with Gamma-Chi-square prior:

$$\theta_{SELF} = E(\theta)$$

$$E(\theta) = \int_0^\infty \theta \boldsymbol{P}(\theta/\boldsymbol{x})\ d\theta \qquad (11)$$

$$= \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1} \int_0^\infty \theta e^{-\theta\beta_1}\ \theta^{\alpha_1 - 1} d\theta$$

$$= \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1} \frac{\Gamma(\alpha_1 + 1)}{\beta_1^{\alpha_1 + 1}}$$

$$= \frac{\alpha_1}{\beta_1} = \frac{(a_1 + n + \frac{c_1}{2} - 1)}{(\sum_{i=1}^{n} x_i + b_1 + \frac{1}{2})} \qquad (12)$$

3.2 Quadratic Loss Function (QLF):

A quadratic loss function is defined as:

$$\lambda(x) = C\,(t - x)^2 \qquad (13)$$

for some constant C; the value of the constant makes no difference to a decision, and can be ignored by setting it equal to 1.

The quadratic loss function can also be defined as

$$L(\theta, \theta_{QLF}) = \left(\frac{\theta - \theta_{QLF}}{\theta}\right)^2 \qquad (14)$$

3.2.1. Bayes estimator under QLF with Gamma – Chi-square Prior:

$$\theta_{QLF} = \frac{E(\theta^{-1})}{E(\theta^{-2})}$$

$$E(\theta^{-1}) = \int_0^\infty \theta^{-1} \boldsymbol{P(\theta/x)}\, d\theta$$

$$E(\theta^{-1}) = \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1} \int_0^\infty \theta^{-1}\, e^{-\theta\beta_1}\, \theta^{\alpha_1-1} d\theta$$

$$E(\theta^{-1}) = \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1} \frac{\Gamma(\alpha_1-1)}{\beta_1^{(\alpha_1-1)}}$$

$$= \frac{\beta_1}{(\alpha_1-1)} \qquad (15)$$

Now

$$E(\theta^{-2}) = \int_0^\infty \theta^{-2}\, e^{-\theta\beta_1}\, \theta^{\alpha_1-1} d\theta$$

$$E(\theta^{-2}) = \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1} \int_0^\infty \theta^{-2}\, e^{-\theta\beta_1}\, \theta^{\alpha_1-1} d\theta$$

$$= \frac{\beta_1^{\alpha_1}}{\Gamma\alpha_1} \frac{\Gamma(\alpha_1-2)}{\beta_1^{(\alpha_1-2)}}$$

$$= \frac{\beta^2{}_1}{(\alpha_1-1)(\alpha_1-2)} \qquad (16)$$

$$\theta_{QLF} = \frac{E(\theta^{-1})}{E(\theta^{-2})}$$

$$= \frac{\beta_1}{(\alpha_1-1)} \frac{(\alpha_1-1)(\alpha_1-2)}{\beta^2{}_1}$$

$$= \frac{(\alpha_1-2)}{\beta_1} = \frac{(a_1+n+\frac{c_1}{2}-3)}{(\sum_{i=1}^{n} x_i+b_1+\frac{1}{2})} \qquad (17)$$

4. POSTERIOR RISKS UNFER DIFFERENT LOSS FUNCTIONS:

4.1 Posterior risk of the Bayes estimator under SELF

$$P(\theta_{SELF}) = E\,(\theta^2) - \{E(\theta)\}^2$$

$$P(\theta_{SELF}) = \frac{\alpha_1(\alpha_1+1)}{\beta_1{}^2} - \frac{\alpha_1{}^2}{\beta_1{}^2}$$

$$= \frac{\alpha_1}{\beta_2{}^2} = \frac{(a_1+n+\frac{c_1}{2}-1)}{(\sum_{i=1}^{n} x_i+b_1+\frac{1}{2})^2} \qquad (18)$$

4.2 Posterior risk of the Bayes estimator under QLF

$$P(\theta_{QLF}) = 1 - \frac{\{E(\theta^{-1})\}^2}{E(\theta^{-2})}$$

$$P(\theta_{QLF}) = 1 - \frac{(\frac{\beta_1}{(\alpha_1-1)})^2}{(\frac{\beta^2{}_1}{(\alpha_1-1)(\alpha_1-2)})}$$

$$= 1 - \frac{(\alpha_1-2)}{(\alpha_1-1)}$$

$$= \frac{1}{(\alpha_1-1)} = \frac{1}{(a_1+n+\frac{c_1}{2}-2)} \qquad (19)$$

5. SIMULATION STUDY:

In this section simulation is carried out to check the performance of double prior. Random samples of sizes n= 10, 20,30,50,75,100 to represent small, medium and large data are generated from Exponential distribution. The scale parameter is estimated for Exponential distribution under double prior selection Gamma-Chi-square with $\theta = 0.5(0.5)3.0$. All the results are carried out using R software.

6. COMPARISON UNDER THE LOSS FUNCTIONS WITH RESPECT TO POSTERIOR RISK:

Posterior risks of the posterior distribution under loss functions is calculated by assuming different sets of hyper parameters 15,25,35,45,55,65,75. Tables I to VI summarizes the posterior risk under different loss function SELF and QLF based on double prior selection Gamma-Chi-square. The posterior risk under SELF is less compared to QLF.

7. CONCLUSIONS:

Posterior risk based on Gamma-Chi-square and for all the loss functions, decreases with increase in sample size. Posterior risk decreases with increase in the value of the parameter θ for different choice of hyper parameters. Loss function under QLF seems to be constant as the value of θ iincreases, which is dependent on the parameter. The loss function SELF has minimum posterior risk compared to QLF. So, Gamma-Chi-square can be used as double prior with minimum posterior risk for estimating the parameter of exponential model.

Table I to VI shows the Posterior Risk of Gamma- Chi-square Double Prior using SELF/ QLF

TABLE-1

Sample size	Hyper para-meters a=b=c	Loss functions	θ =0.5	θ =1	θ = 1.5	θ = 2	θ = 2.5	θ = 3
n=10	15	SELF	0.022681	0.050517	0.059940	0.032667	0.023754	0.018378
		QLF	0.032787	0.032787	0.032786	0.032787	0.032787	0.032789
	25	SELF	0.020813	0.038022	0.042896	0.027591	0.021584	0.017510
		QLF	0.021978	0.021978	0.021978	0.021978	0.021978	0.021978
	35	SELF	0.01875 3	0.030410	0.033379	0.023596	0.019323	0.016313
		QLF	0.016529	0.016529	0.016529	0.016529	0.016529	0.016529
	45	SELF	0.016906	0.025316	0.027312	0.020523	0.017343	0.015006
		QLF	0.013245	0.013245	0.013245	0.013245	0.013245	0.013245
	55	SELF	0.015326	0.021676	0.023109	0.018124	0.015669	0.013809
		QLF	0.011050	0.011050	0.011050	0.011050	0.011049	0.011049
	65	SELF	0.013984	0.018948	0.020027	0.016211	0.014261	0.012748
		QLF	0.009479	0.009479	0.009479	0.009479	0.009479	0.009479
	75	SELF	0.012842	0.016828	0.017669	0.014655	0.013070	0.011817
		QLF	0.008299	0.008299	0.008298	0.008299	0.008299	0.008299

Table II

Sample size	Hyper para-meters a=b=c	Loss functions	θ =0.5	θ =1	θ = 1.5	θ = 2	θ = 2.5	θ = 3
n=20	15	SELF	0.010447	0.046245	0.057516	0.020197	0.012048	0.008633
		QLF	0.024691	0.024691	0.024691	0.024691	0.024691	0.024691
	25	SELF	0.010595	0.035389	0.041582	0.018456	0.011975	0.008977
		QLF	0.018018	0.018018	0.018018	0.018018	0.018018	0.018018
	35	SELF	0.010372	0.028649	0.032559	0.016753	0.011547	0.008959
		QLF	0.014184	0.014184	0.014184	0.014184	0.014184	0.014184
	45	SELF	0.009995	0.024063	0.026753	0.015248	0.010997	0.008766
		QLF	0.011696	0.011696	0.011696	0.011696	0.011696	0.011696
	55	SELF	0.009562	0.020740	0.022705	0.013940	0.010421	0.008491
		QLF	0.009950	0.009950	0.009950	0.009950	0.009950	0.009950
	65	SELF	0.009119	0.018223	0.019720	0.012819	0.009862	0.008181
		QLF	0.008658	0.008658	0.008658	0.008658	0.008658	0.008658
	75	SELF	0.008688	0.016250	0.017428	0.011853	0.009335	0.007861
		QLF	0.007663	0.007663	0.007663	0.007663	0.007663	0.007663

Sample size(n)	Hyper para-meters a=b=c	Loss function s	θ =0.5	θ =1	θ = 1.5	θ = 2	θ = 2.5	θ = 3
n =30	15	SELF	0.008023	0.027485	0.050746	0.012927	0.007328	0.005197
		QLF	0.019802	0.019802	0.019802	0.019802	0.019802	0.019802
	25	SELF	0.008188	0.023419	0.037955	0.012438	0.007553	0.005541
		QLF	0.015267	0.015267	0.015267	0.015267	0.015267	0.015267
	35	SELF	0.008131	0.020348	0.030306	0.011796	0.007559	0.005703
		QLF	0.012422	0.012422	0.012422	0.012422	0.012422	0.012422
	45	SELF	0.007958	0.017967	0.025220	0.011129	0.007447	0.005750
		QLF	0.010471	0.010471	0.010471	0.010471	0.010471	0.010471
	55	SELF	0.007728	0.016074	0.021593	0.010485	0.007271	0.005726
		QLF	0.009050	0.009050	0.009050	0.009050	0.009050	0.009050
	65	SELF	0.007472	0.014536	0.018879	0.009886	0.007063	0.005656
		QLF	0.007968	0.007968	0.007968	0.007968	0.007968	0.007968
	75	SELF	0.007207	0.013264	0.016770	0.009335	0.006840	0.005559
		QLF	0.007117	0.007117	0.007117	0.007117	0.007117	0.007117

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Table III

Table IV

Sample size(n)	Hyper para-meters a=b=c	Loss functions	θ = 0.5	θ = 1	θ = 1.5	θ = 2	θ = 2.5	θ = 3
n=50	15	SELF	0.005567	0.021296	0.029474	0.006374	0.004913	0.003364
		QLF	0.014184	0.014184	0.014184	0.014184	0.014184	0.014184
	25	SELF	0.005687	0.018738	0.024637	0.006438	0.005068	0.003564
		QLF	0.011696	0.011696	0.011696	0.011696	0.011696	0.011696
	35	SELF	0.005710	0.016707	0.021163	0.006402	0.005131	0.003693
		QLF	0.009950	0.009950	0.009950	0.009950	0.009950	0.009950
	45	SELF	0.005671	0.015063	0.018548	0.006307	0.005134	0.003770
		QLF	0.008658	0.008658	0.008658	0.008658	0.008658	0.008658
	55	SELF	0.005593	0.013708	0.016507	0.006177	0.005096	0.003810
		QLF	0.007663	0.007663	0.007663	0.007663	0.007663	0.007663
	65	SELF	0.005492	0.012573	0.014871	0.006027	0.005031	0.003822
		QLF	0.006873	0.006873	0.006873	0.006873	0.006873	0.006873
	75	SELF	0.005376	0.011610	0.013530	0.005867	0.004949	0.003814
		QLF	0.006231	0.006231	0.006231	0.006231	0.006231	0.006231

Sample size(n)	Hyper para-meters a=b=c	Loss functions	θ = 0.5	θ = 1	θ = 1.5	θ = 2	θ = 2.5	θ = 3
n = 75	15	SELF	0.003376	0.013923	0.019626	0.003604	0.002401	0.002038
		QLF	0.010471	0.010471	0.010471	0.010471	0.010471	0.010471
	25	SELF	0.003477	0.012822	0.017370	0.003698	0.002517	0.002153
		QLF	0.009050	0.009050	0.009050	0.009050	0.009050	0.009050
	35	SELF	0.003538	0.011865	0.015576	0.003751	0.002602	0.002241
		QLF	0.007968	0.007968	0.007968	0.007968	0.007968	0.007968
	45	SELF	0.003570	0.011032	0.014116	0.003774	0.002664	0.002308
		QLF	0.007117	0.007117	0.007117	0.007117	0.007117	0.007117
	55	SELF	0.003580	0.010302	0.012906	0.003774	0.002706	0.002359
		QLF	0.006431	0.006431	0.006431	0.006431	0.006431	0.006431
	65	SELF	0.003573	0.009659	0.011886	0.003757	0.002734	0.002395
		QLF	0.005865	0.005865	0.005865	0.005865	0.005865	0.005865
	75	SELF	0.003554	0.009088	0.011015	0.003729	0.002749	0.002420
		QLF	0.005391	0.005391	0.005391	0.005391	0.005391	0.005391

Table VI

Sample size(n)	Hyper para-meters a=b=c	Loss functions	θ = 0.5	θ = 1	θ = 1.5	θ = 2	θ = 2.5	θ = 3
n =100	15	SELF	0.002492	0.009186	0.018571	0.002694	0.001784	0.001343
		QLF	0.008299	0.008299	0.008299	0.008299	0.008299	0.008299
	25	SELF	0.002562	0.008735	0.016525	0.002761	0.001859	0.001413
		QLF	0.007380	0.007380	0.007380	0.007380	0.007380	0.007380
	35	SELF	0.002612	0.008312	0.014885	0.002806	0.001919	0.001472
		QLF	0.006645	0.006645	0.006645	0.006645	0.006645	0.006645
	45	SELF	0.002647	0.007918	0.013541	0.002834	0.001966	0.001521
		QLF	0.006042	0.006042	0.006042	0.006042	0.006042	0.006042
	55	SELF	0.002668	0.007554	0.012420	0.002850	0.002003	0.001562
		QLF	0.005540	0.005540	0.005540	0.005540	0.005540	0.005540
	65	SELF	0.002679	0.007217	0.011470	0.002855	0.002032	0.001597
		QLF	0.005116	0.005116	0.005116	0.005116	0.005116	0.005116
	75	SELF	0.002682	0.006905	0.010656	0.002851	0.002052	0.001625
		QLF	0.004751	0.004751	0.004751	0.004751	0.004751	0.004751

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