Volume : 5 | Issue : 2 | February 2016 • ISSN No 2277 - 8179

Research Paper

Mathematics

**KEYWORDS :** Diffusion Equation- Lie Point symmetric vector field.

# A Study on Similarity Solutions of a Nonlinear Diffusion Equation

**L.Sivakami** | Asst. Prof in Mathematics, SRM University, Kattankulathur

**ABSTRACT**

*Nonlinear diffusion equations, an important class of parabolic appear widely in nature. They are suggested as mathematical models of physical problems in many fields, such as filtration, phase transition, bio chemistry and dynamics of biological groups. In many cases, the equations possess degeneracy or singularity. The appearance of degeneracy or singularity makes the study more involved and challenging. Many new ideas and methods have been developed to overcome the special difficulties caused by the degeneracy and singularity, which enrich the theory of partial differential equations.*

*The motivation for the present study had its origin in my attempt to carry over these techniques, either singly or collectively as the case may be, for obtaining the nonlinear diffusion equation and its symmetry reductions, namely, the second-order nonlinear ordinary differential equations via the isovector approach and further group invariance techniques respectively.*

*The fundamental basis of the techniques is that , when a differential equation in invariant under a Lie group of transformations, a reduction transformation exists.*

*The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions, a single group reduction transforms the partial differential equation into ordinary differential equations which are generally easier to solve than the original partial differential equation.*

*Local symmetries admitted by a nonlinear partial differential equation are useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists.*

*Nonlinear diffusion equations, an important class of parabolic appear widely in nature. They are suggested as mathematical models of physical problems in many fields, such as filtration, phase transition, bio chemistry and dynamics of biological groups. In many cases, the equations possess degeneracy or singularity. The appearance of degeneracy or singularity makes the study more involved and challenging. Many new ideas and methods have been developed to overcome the special difficulties caused by the degeneracy and singularity, which enrich the theory of partial differential equations.*

*The motivation for the present study had its origin in my attempt to carry over these techniques, either singly or collectively as the case may be, for obtaining the nonlinear diffusion equation and its symmetry reductions, namely, the second-order nonlinear ordinary differential equations via the isovector approach and further group invariance techniques respectively.*

*The fundamental basis of the techniques is that , when a differential equation in invariant under a Lie group of transformations, a reduction transformation exists.*

*The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions, a single group reduction transforms the partial differential equation into ordinary differential equations which are generally easier to solve than the original partial differential equation.*

*Local symmetries admitted by a nonlinear partial differential equation are useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists.*

## 1. *Preliminaries*

**Definition 1.1:** Let $\bar{x} = \left(x_1, x_2 .................... x_n\right)$ lie in region $D \subset R^n$.

The set of transformations $x^* = X(\bar{x}, \varepsilon)$, defined for each $\bar{x}$ in D, depending on parameter $\varepsilon$ lying in $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters $\varepsilon$ and $\delta$ in S, forms a group of transformations on D if:

(i) For each parameter $\varepsilon$ in S the transformations are one-to-one, onto D, in particular $x^*$ lies in D.

(ii) S with the law of composition $\phi$ forms a group G.

(iii) $x^* = \bar{x}$ when $\varepsilon = e$, i.e., $X\left(\bar{x}; \varepsilon\right) = \bar{x}$.

(iv) If $x^* = X(\bar{x}; \varepsilon)$, $x^{**} = X\left(x^*; \delta\right)$, then $x^{**} = X\left(\bar{x}; \phi(\varepsilon, \delta)\right)$.

(v) $\varepsilon$ is a continuous parameter, i.e., S is an interval in R. Without loss of generality $\varepsilon=0$ corresponds to the identity element $\varepsilon$.

(vi) X is infinitely differentiable with respect to $\bar{x}$ in D and an analytic function of $\varepsilon$ in S.

(vii) $\phi(\varepsilon,\delta)$ is an analytic function of $\varepsilon$ and $\delta$, $\varepsilon \in S$, $\delta \in S$.

**Definition 1.2**

The infinitesimal generator of the one-parameter Lie group of transformations $x^* = X(\bar{x}, \varepsilon)$ is the operator

$$X = X(\bar{x}) = \xi(\bar{x}).\nabla = \sum_{i=1}^{n} \xi_i(\bar{x}) \frac{\partial}{\partial x_i},$$

where $\nabla$ is the gradient operator, $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\ldots\ldots\ldots \frac{\partial}{\partial x_n} \right)$;

for any differentiable function $F(\bar{x}) = F(x_1, x_2, \ldots\ldots\ldots\ldots x_n)$,

$$XF(\bar{x}) = \xi(\bar{x}).\nabla F(\bar{x}) = \sum_{i=1}^{n} \xi_i(\bar{x}) \frac{\partial F(\bar{x})}{\partial x_i}.$$

**Definition 1.3**

The total derivative operator is defined by

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + y_{n+1} \frac{\partial}{\partial y_n} + \ldots\ldots\ldots$$

For a given differentiable function $F(x, y, y_1, y_2, \ldots\ldots\ldots\ldots\ldots\ldots y_\ell)$, we have

$$\frac{D}{Dx} F(x, y, y_1, y_2, \ldots\ldots\ldots y_\ell) = F_x + y_1 F_y + y_2 F_{y_1} + y_3 F_{y_2} + \ldots\ldots\ldots + y_{\ell+1} F_{y_\ell}.$$

**Definition 1.4**

Let $U=\sum_{i=1}^{p} a^{i}(x,u)\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q} d_{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}$ be a vector field. The n-th prolongation of U is the vector field

$Pr^{(n)}U=U+\sum_{\alpha=1}^{q}\sum_{J} d_{\alpha}^{J}\left(x,u^{(n)}\right)\frac{\partial}{\partial u_{J}^{\alpha}}$, the second summation being over all multi-indices $J=\left(j_1,..........j_k\right)$, with $1\le j_k \le p, \quad 1\le k\le n$.

The coefficient functions $d_{\alpha}^{J}$ of $Pr^{(n)}U$ are given by the following formulas :

$$d_{\alpha}^{J}\left(x,u^{(n)}\right)=D_J\left(d_{\alpha}-\sum_{i=1}^{p} a^{i}u_{i}^{\alpha}\right)+\sum_{i=1}^{p} a^{i}u_{J,i}^{\alpha},$$

where $u_{i}^{\alpha}=\frac{\partial u^{\alpha}}{\partial x^{i}}$, and $u_{J,i}^{\alpha}=\frac{\partial u_{J}^{\alpha}}{\partial x^{i}}$

and $$d_{\alpha}^{J,k}=D_K D_J\left(d_{\alpha}-\sum_{i=1}^{p} a^{i}u_{i}^{\alpha}\right)+\sum_{i=1}^{p} a^{i}u_{J,ik}^{\alpha},$$

where $u_{J,ik}^{\alpha}=\frac{\partial^{2}u_{J}^{\alpha}}{\partial x^{i}\partial x^{k}}$.

**Definition 1.5**

Consider an r-parameter Lie group of transformations $x^{*}=X(\bar{x};\,\varepsilon)$ with infinitesimal generators $\{X_{\alpha}\}, \alpha=1,2,..........r$, defined by

$$\xi_{\alpha j}(\bar{x})=\left.\frac{\partial x_{j}^{*}}{\partial \varepsilon_{\alpha}}\right|_{\varepsilon=0}=\left.\frac{\partial X_{j}(\bar{x};\varepsilon)}{\partial \varepsilon_{\alpha}}\right|_{\varepsilon=0} \qquad \alpha=1,2,........r,\ j=1,2,......n.$$

and

$$X_\alpha = \sum_{j=1}^{n} \xi_{\alpha j}\left(\bar{x}\right)\frac{\partial}{\partial x_j}, \quad \alpha=1,2,........r.$$

The commutator of $X_\alpha$ and $X_\beta$ is a first-order operator

$$\left[X_\alpha, X_\beta\right] = X_\alpha X_\beta - X_\beta X_\alpha = \sum_{i,j=1}^{n}\left[\begin{array}{c}\left(\xi_{\alpha i}(x)\frac{\partial}{\partial x_i}\right)\left(\xi_{\beta j}(x)\frac{\partial}{\partial x_j}\right) \\ -\left(\xi_{\beta i}(x)\frac{\partial}{\partial x_i}\right)\left(\xi_{\alpha j}(x)\frac{\partial}{\partial x_j}\right)\end{array}\right]$$

$$= \sum_{i,j=1}^{n} \eta_j(x)\frac{\partial}{\partial x_j},$$

where

$$\eta_j(x) = \sum_{i=1}^{n}\left(\xi_{\alpha i}(x)\frac{\partial \xi_{\beta j}(x)}{\partial x_i} - \xi_{\beta i}(x)\frac{\partial \xi_{\alpha j}(x)}{\partial x_i}\right).$$

It immediately follows that

$$\left[X_\alpha, X_\beta\right] = -\left[X_\beta, X_\alpha\right].$$

**Definition 1.6**

A Lie algebra ₤ is a vector space over some field ₣ with an additional law of combination of elements in ₤ (the commutator) satisfying the properties

$$\left[X_\alpha, X_\beta\right] = -\left[X_\beta, X_\alpha\right]$$

$$\left[X_\alpha, \left[X_\beta, X_\gamma\right]\right] + \left[X_\beta, \left[X_\gamma, X_\alpha\right]\right] + \left[X_\gamma, \left[X_\alpha, X_\beta\right]\right] = 0$$

with most importantly, closure with respect to commutation.

In particular the infinitesimal generators $\{X_\alpha\}, \quad \alpha=1,2,.....r,$

of an r-parameter Lie group of transformations $x^{*} = X(\bar{x};t)$ form an r-dimensional Lie algebra $£^r$ over (the field) R since for any

$X_\alpha, X_\beta, X_\gamma \in £^r$, ,a,b $\in R$:

(i) $aX_\alpha + bX_\beta \in £^r$;

(ii) $X_\alpha + X_\beta = X_\beta + X_\alpha$;

(iii) $X_\alpha + \left(X_\beta + X_\gamma\right) = \left(X_\alpha + X_\beta\right) + X_\gamma$;

(iv) $\left[X_\alpha, X_\beta\right] \in £^r$;

(v) $\left[X_\alpha, X_\beta\right] = -\left[X_\beta, X_\alpha\right]$;

(vi) $\left[X_\alpha, \left[X_\beta, X_\gamma\right]\right] + \left[X_\beta, \left[X_\gamma, X_\alpha\right]\right] + \left[X_\gamma, \left[X_\alpha, X_\beta\right]\right] = 0$;

(vii) $\left[aX_\alpha + bX_\beta, X_\gamma\right] = a\left[X_\alpha, X_\gamma\right] + b\left[X_\beta, X_\gamma\right]$.

## SIMILARITY SOLUTION OF THE INHOMOGENEOUS NONLINEAR DIFFUSION EQUATION

Here it is concerned with some enlargements of the similarity re duction of the inhomogeneous nonlinear diffusion equation

$$x^p \frac{\partial u}{\partial t}(x,y,t) = \frac{\partial}{\partial x}\left(x^m u^n u_x\right) + \lambda \frac{\partial}{\partial y}\left(y^\ell u^q u_y\right), \qquad (2.1)$$

where p,q,$l$,m and n are arbitrary constants , $\lambda$ is a parameter. First we determine the Lie point symmetry vector fields.

Let

$$U = a(x,y,t,u)\frac{\partial}{\partial x} + b(x,y,t,u)\frac{\partial}{\partial y} + c(x,y,t,u)\frac{\partial}{\partial t} + d(x,y,t,u)\frac{\partial}{\partial u}, \qquad (2.2)$$

where a,b,c and d are unspecified functions of x,y,t and u. We apply the algorithm that provides the symmetry algebra by constructing the prolongation of the vector field U.

$$Pr^2U=U+d^x\partial_{u_x}+d^y\partial_{u_y}+d^t\partial_{u_t}+d^{xx}\partial_{u_{xx}}+d^{yy}\partial_{u_{yy}}, \qquad (2.3)$$

where

$$d^x=d_x+(d_u-a_x)u_x-a_uu_x^2-b_xu_y-b_uu_xy_y-c_xu_t-c_uu_xu_t \qquad (2.4)$$

$$d^y=d_y+(d_u-b_y)u_y-a_yu_x-a_uu_xu_y-b_uu_y^2-c_yu_t-c_uu_yu_t, \qquad (2.5)$$

$$d^t=d_t+(d_u-c_t)u_t-a_tu_x-a_uu_xu_t-b_tu_y-b_uu_tu_y-c_uu_t^2, \qquad (2.6)$$

$$\begin{aligned}d^{xx}=&\,d_{xx}+(2d_{ux}-a_{xx})u_x+u_{xx}(d_u-2a_x)-c_{xx}u_t\\&+u_x^2(d_{uu}-2a_{xu})-3a_uu_xu_{xx}-2c_{xu}u_xu_t-a_{uu}u_x^3\\&-c_{uu}u_x^2u_t-2b_xu_{xy}-2b_uu_xu_{xy}-b_{xx}u_y-2b_{xu}u_xu_y\\&-b_{uu}u_x^2u_y-b_uu_{xx}u_y-2c_xu_{xt}-2c_uu_xu_{xt}-c_uu_{xx}u_t,\end{aligned} \qquad (2.7)$$

and

$$\begin{aligned}d^{yy}=&\,d_{yy}+(2d_{uy}-b_{yy})u_y+u_{yy}(d_u-2b_y)-c_{yy}u_t\\&+u_y^2(d_{uu}-2b_{uy})-3b_uu_yu_{yy}-2c_{uy}u_yu_t-b_{uu}u_y^3-c_{uu}u_y^2u_t\\&-2a_yu_{xy}-2a_uu_yu_{xy}-a_{yy}u_x-2a_{uy}u_xu_y-a_{uu}u_y^2u_x\\&-a_uu_{yy}u_x-2c_yu_{yt}-2c_uu_yu_{yt}-c_uu_{yy}u_t.\end{aligned} \qquad (2.8)$$

The condition of invariance of the equation (2.1) is

$$Pr^2U(\Delta)\big|_{\Delta=0}=0, \qquad (2.9)$$

where $\Delta=u_t-mx^{m-1-p}u^nu_x-nu^{n-1}x^{m-p}u_x^2-x^{m-p}u^nu_{xx}$

$$-\lambda ly^{\ell-1}u^qu_yx^{-p}-\lambda qu^{q-1}y^{\ell}u_y^2x^{-p}-\lambda y^{\ell}u^qu_{yy}x^{-p}.$$

From (2.3) and (2.9) we get,

$$a\begin{pmatrix} -m(m-1-p)x^{m-2-p}u^{n}u_{x}-n(m-p)\,u^{n-1}\,x^{m-p-1}u_{x}^{2}-(m-p)x^{m-p-1}u^{n}u_{xx} \\ +p\lambda\ell y^{\ell-1}u^{q}u_{y}x^{-p-1}+p\lambda qu^{q-1}y^{\ell}u_{y}^{2}x^{-p-1}+p\lambda y^{\ell}u^{q}u_{yy}x^{-p-1} \end{pmatrix}$$

$$+b\left(-\lambda\ell(\ell-1)y^{\ell-2}u^{q}u_{y}x^{-p}-\lambda q\ell u^{q-1}y^{\ell-1}u_{y}^{2}x^{-p}-\lambda\ell y^{\ell-1}u^{q}u_{yy}x^{-p}\right)$$

$$+d\begin{pmatrix} -mnx^{m-1-p}u^{n-1}u_{x}-n(n-1)u^{n-2}x^{m-p}u_{x}^{2}-nx^{m-p}u^{n-1}u_{xx} \\ -\lambda q\ell y^{\ell-1}u^{q-1}u_{y}x^{-p}-\lambda q(q-1)u^{q-2}y^{\ell}u_{y}^{2}x^{-p}-\lambda qy^{\ell}u^{q-1}u_{yy}x^{-p} \end{pmatrix}$$

$$+d^{x}\left(-mx^{m-1-p}u^{n}-2nu^{n-1}x^{m-p}u_{x}\right)+d^{y}\left(-\lambda ly^{\ell-1}u^{q}x^{-p}-2\lambda qu^{q-1}y^{\ell}u_{y}x^{-p}\right)$$

$$+d^{t}+d^{xx}\left(-x^{m-p}u^{n}\right)+d^{yy}\left(-\lambda y^{\ell}u^{q}x^{-p}\right)=0.$$

Using (2.4) –(2.8) and (2.9) in the above equation we get the following set of determining equations

$$a_{xx}-\frac{ma_{x}}{x}+\frac{am}{x^{2}}=0, \tag{4.10}$$

$$\frac{nd}{u^{2}}-\frac{nd_{u}}{u}-d_{uu}=0, \tag{4.11}$$

$$-\frac{a(m-p)}{x}-\frac{dn}{u}+2a_{x}-c_{t}=0, \tag{4.12}$$

$$\frac{2q}{u}b_{y}+\frac{q}{u}d_{u}-d_{uu}+\frac{dq}{u^{2}}=0, \tag{4.13}$$

$$\frac{2d_u}{y}+\frac{b}{y^2}+\frac{b_y}{y}+\ell b_{yy}=0. \tag{4.14}$$

Solve the equations (4.10) –(4.14), we get

$$a=\frac{2c_2}{p-m+2}\ x-\frac{nc_1}{p-m+2}x,$$

$$b=c_1y,$$

$$c=2c_2t+c_3,$$

$$d=-c_1u,$$

where $c_1$, $c_2$ , $c_3$ are arbitrary constants and $p-m+2=v$.

Then we have the three symmetry vector fields,

$$B_1=-\frac{nx}{v}\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}+y\frac{\partial}{\partial y},$$

$$B_2=2t\frac{\partial}{\partial t}+\frac{2x}{v}\ \frac{\partial}{\partial x},$$

$$B_3=\frac{\partial}{\partial t}.$$

Now,

$$\left[B_1, B_1\right]=B_1^2-B_1^2=0,$$

$$\left[B_1, B_2\right]=B_1B_2-B_2B_1$$

$$=\left(\frac{-nx}{v}\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}+y\frac{\partial}{\partial y}\right)\left(2t\frac{\partial}{\partial t}+\frac{2x}{v}\frac{\partial}{\partial x}\right)$$

$$-\left(2t\frac{\partial}{\partial t}+\frac{2x}{v}\frac{\partial}{\partial x}\right)\left(\frac{-nx}{v}\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}+y\frac{\partial}{\partial y}\right)$$

$$=\left(\frac{-nx}{v}\right)\frac{2}{v}\frac{\partial}{\partial x}-\left(\frac{nx}{v}\right)\frac{2x}{v}\frac{\partial^2}{\partial x^2}$$

$$+\left(\frac{2x}{v}\right)\left(\frac{n}{v}\right)\frac{\partial}{\partial x}+\left(\frac{2x}{v}\right)\left(\frac{nx}{v}\right)\frac{\partial^2}{\partial x^2}$$

$$\text{i.e.,}\left[B_1, B_2\right]=0,$$

$$\left[B_1, B_3\right]=B_1B_3\text{-}B_3B_1$$

$$=\left(\frac{-nx}{v}\frac{\partial}{\partial x}\text{-}u\frac{\partial}{\partial u}+y\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t}\right)-\left(\frac{\partial}{\partial t}\right)\left(\frac{-nx}{v}\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}+y\frac{\partial}{\partial y}\right).$$

$$\text{i.e.,}\left[B_1, B_3\right]=0,$$

$$\left[B_2, B_1\right]=-\left[B_1, B_2\right]=0,$$

$$\left[B_2, B_2\right]= B_2^2 - B_2^2 =0,$$

$$\left[B_2, B_3\right]=B_2B_3\text{-}B_3B_2$$

$$=\left(2t\frac{\partial}{\partial t}+\frac{2x}{v}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right)\text{-}\left(\frac{\partial}{\partial t}\right)\left(2t\frac{\partial}{\partial t}+\frac{2x}{v}\frac{\partial}{\partial x}\right)$$

$$=2t\frac{\partial^2}{\partial t^2}\text{-}2\frac{\partial}{\partial t}-2t\frac{\partial^2}{\partial t^2}$$

$$\text{i.e.,}\left[B_2, B_3\right]=-2B_3,$$

$$\left[B_3, B_1\right]=-\left[B_1, B_3\right]=0,$$

$$\left[B_3, B_2\right]=\text{-}\left[B_2, B_3\right]=2B_3,$$

$$\left[B_3, B_3\right]=0.$$

Volume : 5 | Issue : 2 | February 2016 • ISSN No 2277 - 8179

These field form a Lie algebra. We find $\left[B_2, B_3\right] = -2B_3$, and all other commutations vanish.

**For the vector field $B_1$**

The characteristic equations are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dt}{c} = \frac{du}{d}$$

i.e., $$\frac{dx}{\frac{-nx}{v}} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{-u}.$$

Then

$$\frac{u}{x^{\frac{v}{n}}} = c_1,$$

$$t = c_2,$$

and $$yx^{\frac{v}{n}} = c_3,$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

The general integral is $$F\left(yx^{\frac{v}{n}}, t, \frac{u}{x^{\frac{v}{n}}}\right) = 0,$$

where F is an arbitrary function.

Then the similarity solution and the Similarity variable are given by

$$u = x^{\frac{v}{n}} F(s), \qquad s = t + yx^{\frac{v}{n}}.$$

**CONCLUSION:** In this paper I have studied the solutions of nonlinear diffusion equations by applying Lie symmetry method and derived the similarity solution of the inhomogeneous nonlinear diffusion equation .

## REFERENCE

[1] Barenbatt G.I., Similarity, Self-Similarity, and Intermediate Asymptotics, New York: Consultants Bureau, 1979. | [2] Berryman J.G., J.Math. Phys., 21 (1980), 1326-1331. | [3] Bluman G.W. and Cole J.D., Similarity Methods for Differential Equations, Berlin, Springer, 1974. | [4] Grundy R.E., Q.Appl. Math.,37(1979), 259-280. | [5] Hill J.M., J.Eng. Math., 23(1989), 141-155. | [6] Hill D.L. and Hill J.M., J.Eng. Math., 24(1990), 109-124. | [7] Hydun P.E., Symmetry Methods for Differential Equations, | Cambridge University Press, Cambridge, 2000. | [8] King J.R., IMA J.Appl.Math., 40(1988), 163-181. | [9] King J.R., Q.J.Mech. Appl. Math., 42(1989), 537-552. | [10] Lacey A.A., Ockendon J.R. and Tayler A.B., SIAM J.Appl.Math., | 42(1982), 1252-1264. | [11] Olver P.J., Application of Lie Groups to Differential Equations, | Berlin, Springer, 1986. | [12] Olver P.J. and Rosenau P., Phys.Lett.,A 114(1986), 107-112; SIAM J.Appl.Math.,47(1987), 263-278. |