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Research Paper

# Dese Differential Algebras of Generalized Functions

Mathematics

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## ABSTRACT

*We develop the basic idea of a sheaf, look at some simple examples and explore areas of mathematics which become more transparent and easier to think about in light of this new concept. Though we attempt to avoid being too dependent on category theory and homological algebra, a reliance on the basic language of the subject is inevitable when we start discussing sheaf cohomology.*

**Introduction**

Differential geometry is recent extension of classical differential geometry on smooth manifolds which, however, does no longer use any notion of calculus. Instead of smooth functions, one start with sheaf of algebras i.e. the structure sheaf, considered on an arbitray topological space, which is the base space of all the sheaves subsequently involved. Since it places a powerful apparatus at our disposal which can reproduce and therefore extend fundamental classical results. The aim is to give an indication of the extent to which this apparatus can go beyond the classical frame work by including the largest class of singularities dealt with so far. Thus it is shown that instead of classical structure sheaf of algebras of smooth functions, one can start with a significantly larger, and non smooth sheaf of so called Dense Differential Algebras of generalized functions. In the course of our study it often becomes necessary to restrict our study how these ideas apply in a more specialized area. Sometimes they might apply in an interesting way, sometimes in a trivial way, but they will always apply in some way. While doing mathematics, we might notice that two ideas which seemed disparate on a general scale exhibit very similar behavior when applied to more specific examples, and eventually be led to deduce that their overarching essence is in fact the same. Alternatively, we might identify similar ideas in several different areas, and after some work and that we can indeed `glue them together' to get a more general theory that encompasses them all. Infancy terminology, mathematics is a bit like a sheaf.

Definition . A presheaf F of abelian groups on a topological space X is an object associating with every open set U ⊂ X an abelian group F (U) and with every inclusion U ⊂ V of two open sets in X a group homomorphism

⊂VU: F(V ) → F(U). Furthermore, ⊂UU must be the identity mapping and if

U ⊂ V ⊂ W then ⊂WU = ⊂VU ⊂ ⊂WV.

The group F (U) is called the group of sections over U, and the morphism ⊂VU is called the restriction map of F(V ) onto F(U).

Definition. A sheaf F is a presheaf that satisfies the following condition.

Let {Ui}iI be any collection of open sets on X and U = i.

Then:

1. (Existence of gluing) Whenever, for every iI, si (Ui) and for every

i, j I,si| Ui⊂Uj = sj| Ui⊂Uj there exists a section s F(U) with s|Ui = si

for all i I.

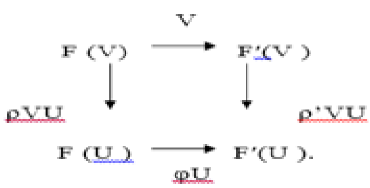
2. (Uniqueness of gluing) Let s, t F (U ) be such that s|Ui = t|Ui for all i I. Then s = t.

**Morphisms of sheaves**

Firstly, having de ned sheaves of abelian groups as generalizations of (fixed) abelian groups, it is natural to want to de ne morphisms between them in order to compare different sheaves. This aspiration becomes rather complex and unnatural unless the underlying topological space is the same for both the domain and codomain of the morphism. So how will we de ne a morphism of sheaves ⊂ : F → F⊂? Well, normal abelian groups can be viewed as sheaves on an indiscrete topology, and their morphisms consist of a single morphism of abelian groups. Now, in this more general situation, we have a whole collection of abelian groups, one for each open set, so it seems natural that for every open set U we will need a morphism of abelian groups

⊂U: F (U ) →F′(U). Is this sucient? Well, no, not quite. Sheaves also have extra structure in the form of restriction maps, and morphisms always have to pay due respect to structure. These considerations put together give rise to the following final definition.

**Definition.** A morphism of presheaves of abelian groups on X, ϕ: F → F′, consists of a morphism ϕU: F(U) → F′(U) of abelian groups for each open set U , and these morphisms must have the property that for any pair of open sets U V , the following diagram commutes:




**A morphism of sheaves is defined in precisely the same way.**

**Manifolds and schemes**

Now briefly describe two areas in which applying a sheaf-theoretic approach to a previously studied locally ringed space has led to rich and important new ideas and generalisations. We will look at differentiable manifolds, and how this concept admits generalisations, as well as how cotangent spaces can be defined abstractly within a locally ringed space. Then we will see how these ideas have been applied to algebraic geometry, sketching the ideas behind Alexander Grothendieck's revolutionary generalisation of zero sets of polynomials to abstract schemes. In fact, both of these concepts are, in light of our new sheaf-based formalism, closely related.

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Firstly, let us take a look at n-dimensional differentiable manifolds with fresh eyes. Recall that they are topological spaces on which it is possible to do differentiation because they are locally homeomorphic to a subset of $R^n$. In the language of locally-ringed spaces, we can proceed as follows. Let ($R^n$,O) be n-dimensional space equipped with the standard sheaf of rings of smooth functions (familiar from any undergraduate analysis course - note how the openness of the sets on which sections of a sheaf are defined becomes important). This is a locally ringed space, and defining a manifold is now easy with our new language.

**Definition**. A smoothly differentiable n-dimensional manifold is any locally ringed space (M, $O_M$) which is locally isomorphic to a subset of ($R^n$, O).

In other words, for any x M there exists a neighbourhood U of x such that some map f: U → $R^n$ induces an isomorphism

(U, $O_{M|U}$) = (f(U), $O|_{f(U)}$) of locally ringed spaces. It is fairly straightforward to check that this definition is equivalent to the usual one with charts and atlases and so on. What is nice is that in this definition we really get to the heart of the matter of what a manifold is. It's not really some nonsense about gluing together a few coordinate systems - the property of interest is that it is locally isomorphic to euclidean space, at least for the purposes of doing differentiation. There are also some practical uses. For example, we now have suggested to us a rather nice abstract definition of the cotangent space. Intuitively this space is the space of `first order infinitesimals' around a function (or, in the language of the next section, the space of local 1-forms about a point). Equipped with the formalism of local rings, we can rigorously express infinitessimals as equivalence classes of functions which all head off in the same direction, to first order. Let Ox be the local ring at x, and Mx its maximal ideal. Since we care about infinitessimals, all our cotangents will be elements of Mx (i.e. they are zero, but might be going somewhere nonzero). However, being first order approximations, two elements of Mx will correspond to the same cotangent precisely whenever their difference is in $M^2_x$:= {fg: f,gM}. Hence we obtain a purely abstract definition of the cotangent space as Mx/ $M^2_x$. It can be quickly checked that this is a vector space over the field Ox/Mx, and indeed in smooth manifolds this coincides with the definition of cotangents as dual to the tangent space of derivation operations.

Above all, it must be becoming clear to the reader that many of these familiar geometrical concepts are actually much more general than one might expect. The previous paragraph allows us to define a cotangent space on any locally ringed space and operate on it in analogy with a differentiable manifold. More profoundly however, there was absolutely nothing special about the space ($R^n$, O) as our choice of model in the definition of a manifold. Indeed, replacing R with C and taking a sheaf of analytic functions, we get complex manifolds. A similar much more radical substitution in to this sheaf-theoretic construction of a manifold allows us to define a scheme, an unbelievably deep and useful generalization of the concept of variety (essentially, the set of points in a vector space which are the common zeroes of a fixed system of algebraic equations) from classical algebraic geometry.

Rather than a vector space with some differentiation happening, the underlying structure here is the spectrum of a ring A whose definition is rather involved. Firstly let Spece A be the set of prime ideals of A. For any ideal a of A,

letV(a) be the set of prime ideals which contain a. Take the sets V(a) ⊂ SpecA to be the closed sets generating a topology on SpecA (it can be checked that this procedure works). We now have a nice topological space of prime ideals, whose topology is analogous to the Zariski topology in algebraic spaces.

Next, define O a sheaf of rings on SpecA as follows. For pSpecA let Ap = A/p, and we can now de ne O(U) to be the set of functions

s: U→such that s(p)Ap for all p and s is locally a quotient of elements of A. The local nature of this definition guarantees that this O, which is manifestly a presheaf, is indeed a sheaf, and it generalises the sheaves of locally rational functions one would get on a more concrete algebraic space. Note that the stalks of this sheaf are just Ap but also, perhaps surprisingly, the ring of global sections O(SpecA) is canonically isomorphic to A.

Anyway, we have now made the spectrum of a ring into a locally ringed space (SpecA, O), and can now define a scheme in exactly the same manner as our definition of a manifold.

**Definition**. A scheme is a locally ringed space (X, $O_X$) which is locally isomorphic to the spectrum of a ring (which need not necessarily be the same ring in every neighbourhood).

Of course, the schemes are sim ply a sub category of the category of locally ringed spaces (so we have already defined scheme morphisms).

It is well beyond the scope of this essay to go any further into the huge subject of algebraic geometry, but it is important to realise how absolutely vital the theory of sheaves is for defining locally ringed spaces and in turn for defining these schemes.

Sheaves in algebraic topology: steps towards unifying cohomology

We will start to explore one of the most important applications of sheaf theory in providing a new cohomology whose definition makes it very easy to compare with the cornucopia of other cohomologies that have arisen in algebraic topology over its long and energetic history. Aware that some readers might not be totally happy with what a cohomology even is, making something quite scary a lot scarier in order to make it less scary might have been a more appropriate title. Though it will start quite gently this chapter will be a narrative (omitting most of the quite technical proofs) of some actually very serious and involved 20th century mathematics, and will therefore of necessity be rather faster-paced than the rest of the essay. That said, we hope that even if a reader does not take away a completely clear picture of how they might go about proving these results, they will at least get a flavor of the power of the sheaf-theoretic approach in dealing with the very practical issue of comparing different cohomological theories.

Cohomology of functors and the global sections functor

Our first stage towards developing a cohomology of sheaves will be to set up a general categorical construction called the `derived factor'. Though this will mean doing things in much more generality than we care about, it will make it clearer exactly what we are doing without the clutter of sheaf notation everywhere. We will then apply this construction to the global sections functor Γ: F → F(X) which takes a sheaf of rings (or modules) to its ring (or module) of global sections to obtain the sheaf cohomology. Firstly, a few key definitions of category theory and homological algebra.

**Definition**. A (covariant) functor F : A → B is a mapping of categories that takes objects[2] in A to objects in B and morphisms between objects in A to morphisms between the corresponding objects in B. It must respect identity morphisms and composition of morphisms.

If all the categories in sight are abelian, we say that F is exact if whenever

$0 \to A \to B \to C \to 0$

is a short exact sequence, then the following sequence (with morphisms also transformed under F ) is also exact:

$0 \to F(A) \to F(B) \to F(C) \to 0.$

We say that F is left-exact if, everything as above, we only have that

$0 \to F(A) \to F(B) \to F(C)$

is exact. In other words, if it preserves (in some sense) injections but not necessarily surjections.

If talk of objects and morphisms scares you it should not. An object is just something like a ring, and in all categories we will work with morphism is just an appropriate map between two objects, in our case probably a ring homomorphism. Hence an example of a functor might be the so-called `forgetful' functor from the category of groups from the category of sets, which takes a group to the set containing its elements, and the morphisms to the corresponding maps of sets - it just `forgets' the group structure. Another functor in the other direction is the `free group' functor which makes a free group out of any set, and a mapping of sets is mapped to a group homomorphism determined by the corresponding substitution of the generators.

Once the reader has finished scratching their head (which may take some time), they might guess where we are going next. Just as with the1-forms in the differential equations example, here we will not be content to just say that a functor is left-exact (i.e. an exact functor which has failed to preserve surjections). We will be interested in some kind of measure of to what extent the functor is not exact. The obvious thing to try is just plugging in some exact sequences and measuring how non-surjective the maps $F(B) \to F(C)$ get. It turns out that such a process is just too unwieldy, and ends up with something which is a function of at least two objects in the category. We try something slightly more subtle, and though at first it looks hopeless, we are saved by the presence of a miraculous subcategory called the `injectives.'

Regardless of our procedure, we will always have to pick some object to kick things o, let's call it A. We then construct a (possibly in finitely long never hitting zero) exact complex

$0 \to A \to A^0 \to A^1 \to \ldots..$

If we apply F to this as it stands, we will get exactness at the first two stages, and then some other stu will happen which is basically independent of A. Since we therefore don't care about these first two stages, it is usual to instead cut out the A and consider the composition of the two relevant morphisms as a single map, so we will measure the in exactness of the functor just by taking cohomology groups of the complex (preserving F (A) as the 0th cohomology group, by left-exactness).

$0 \to F(A^0) \to F(A^1) \to F(A^2) \to \ldots$

This still gives us all sorts of answers, and by sticking extra objects in or moving things about we can get all sorts of different cohomologies in this way, even for Axed. The procedure is tantalisingly close to being correct though. The main reason it fails is that once you know a functor doesn't tend to preserve a certain surjection, you can go sticking copies of that surjection all over your A• complex, and immediately alter the higher cohomology. This seems hopeless... the only thing that could save us would be some kind of incredibly well-behaved subcategory of objects restricted to which F is in fact an exact functor, but one's gut (certainly my poorly trained gut) despairs that this might have to be different for different functors (or even starting objects!) and so making this idea work will be a nightmare. However, unbelievably, such a subcategory can at least be defined.

**Defininition.** We say that an object I is an injective if for any injection

$\phi: A \to B$ if there is a map $\theta : A \to I$ it can be extended to $\theta' : B \to I$ in a way that respects the injection ($\theta' \circ \phi = \theta$). Of course, just because we can define it doesn't mean it exists. There might not actually be any injectives out there, or not enough to be useful. Fortunately, the following proposition tells us that in the categories we will be studying there are.

Proposition: For any sheaf of rings O, the category of O-modules has enough injectives: the property that for any object A, there exists an injective I such that A can be embedded in I (i.e. $0 \to A \to I$ is exact).

What makes injectives work so well is that their defining property allows you to compare two different complexes $0 \to A \to A\bullet$ all of whose elements (after A) are injectives by starting with the identity morphism both ways between A and itself, and then extending to a direct comparison both ways of the first pair of injectives using the definition of an injective, and so on. These comparisons turn out to prove that whichever complex of injectives we use to make $0 \to A \to A\bullet$ exact, the complex $0 \to F(A\bullet)$ will always have the same cohomology groups! Therefore we can construct a new set of functors called the right-derived functors of F which measure the extent to which F fails to be exact as a function of A, and we now know the procedure. Take any complex of injectives A• such that $0 \to A \to A0 \to A1 \to \ldots$ is exact (this is possible because there are enough injectives). By the above discussion, the following is well-defined.

**Definition.** Define the right derived functors by $R^nF(A) := H^n(F(A\bullet))$.

Now we have the right derived functors, we have a reasonable measure of how removing a factor of A from an otherwise perfectly well-behaved sequence (on which F is exact) causes it the exactness of the sequence to break up under Fat the various stages down the line. The specialinjectives were vital in building such a `well behaved sequence'. However, surely any other object whose surjections F preserves would have done just as good a job - such an object would itself have caused no damage to the exactness. This turns out to be the case, which makes the derived functors much easier to calculate. This is the F -dependent category of my previous nightmares, but with the help of our wonderful injectives to get us going, we now have a beautiful definition for it.

**Definition.** An object B is F -acyclic if $R^nF(B)$ is trivial for all $n > 0$. As we predict above, the following theorem turns out to be true, and the proof is basically the same as the proof that a choice of complex of injectives does not affect the cohomology there.

Theorem. Let A be any object, and B• be a complex of F -acyclic objects such that

$0 \to A \to B^0 \to B^1 \to B^2 \to \ldots$

is exact. Then

$R^nF(A) = H^n(F(B\bullet)).$

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Ok, so we've now done even more abstract nonsense which seemed to have little to do with sheaves. However, now we are going to apply it to the global sections functor. This is the functor that basically ignores all of the sheaf theory except the `layer at the top' for both objects and morphisms. It is easy to check that this functor is left-exact, but that it may often fail to be actually exact. Recall example of the sheaf of holomorphic functions on a compact Riemann surface having very few global sections relative to its diverse local structure. Phenomena like that are precisely what the derived functors will be able to give us some indications of - sheaves which, though locally exact, deviate from global exactness as the global sections which local sections expect to exist simply fail to because of nontrivial global structure.

With all the machinery above developed, we can go right ahead and define the sheaf cohomology.

Definition 10. Let $\Gamma(X, -): F \to F(X)$, $u \to u_X$ be the global sections functor over X ( from sheaves of modules to modules). Define the sheaf cohomology by

$$H^n(X,F) := R^n\Gamma(X,F).$$

**Conclusion**

In this paper we have gently introduced the idea of a sheaf in mathematics, and hopefully persuaded them that the language and formalism of sheaves, once fully understood, is very powerful and of interest in all sorts of areas, particularly geometry and topology. In particular, we saw how sheaves were important in defining abstract geometrical spaces, giving us a new angle on manifolds, and proving absolutely vital for defining schemes. We then took a much deeper application of sheaf theory in providing a general mechanism (the absence of global sections in exact sequences of sheaves) which surprisingly gives rise to many of the major cohomological theories and sometimes shows naturally where theories are superficially dissimilar how they are related. Providing an easy means of calculating sheaf cohomology in more abstract situations where the notion of cohomology is still useful but no concrete definition available.

**REFERENCES**

Below I detail books and articles of which I have used hard copies. However, I have also fleetingly made use of very many internet resources, but very few in sufficient detail or substance to have kept track of

[1] A. Dimca: Sheaves in Topology.
[2] R. Hartshorne: Algebraic Geometry.
[3] G. Chenevert: Sheaf Cohomology.
[4] G.E. Bredon: Sheaf Theory.
[5] S. Lang: Algebra (Revised 3rd Edition).
[6] F. Borceaux: Handbook of Categorical Algebra.
[7] http://wikipedia.org .
[8] B.R. Tennison: Sheaf Theory.