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DERIVATIONS ACTING AS 3 - HOMOMORPHISMS & 3 - ANTIHOMOMORPHISMS ON PRIME RINGS

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ABSTRACT

Let R be a ring. An additive mapping f:R → R is called 3-homomorphism (resp.3-antihomomorphsm) on R if f(xyz) = f(x)f(y)f(z) (resp. f(xyz) = f(z)f(y)f(x)) for all x, y, z ∈ R. In the present paper, we investigate the form of derivation on a prime ring that acts as a 3 homomorphism or as a 3-antihomorphism on a ring R.

INTRODUCTION:

Throughout this article, R will represent an associative ring with centre Z(R). We denote [x, y] = xy - yx, the commutator of x and y. A ring R is said to be 2-torsion free if 2a = 0 (where a∈R) implies a = 0. A ring R is called a prime ring if aRb = (0) (where a, b R) implies a = 0 or b = 0. An additive mapping f:R→R is called a homomorphism (resp. antihomomorphism) of R if f(xy) = f(x)f(y) (resp. f(xy) = f(y)f(x)) for all x, y ∈R. Following [4], an additive mapping f:R →R is called a n-homomorphism (resp. n-antihomomorphism) of R if

$$f\left(\prod_{i=1}^{n} a_i\right) = \prod_{i=1}^{n} f(a_i)$$

$$\left(\text{resp.} f\left(\prod_{i=1}^{n} a_i\right) = \prod_{i=n}^{1} f(a_i)\right) \text{ for all } a_1, a_2, \ldots.., a_n \in R.$$

For n = 3, it is known as 3-homomorphism (resp. 3-antihomomorphism) of R. The concept of n-homomorphism was studied for complex algebras by Hejazian, Mirzavaziri and Moslehian [4] (see also [5]), where some of their significant properties are investigated on Banach algebras. It can be easily seen that every homomorphism of R is the n-homomorphism of R for all n≥2, but the converse is not true in general. For instance, let f : R→R be a nonzero homomorphism on R. Then -f is a 3-homomorphism on R, but not the homomorphism of R. Some other examples can be found in [4]. Thus, it is reasonable to extend the results for homomorphism and antihomomorphism to n-homomorphism and n-antihomomorphism, respectively. In this paper, we describe the form of generalized derivation which acts as 3-homomorphism and 3-antihomomorphism on prime rings.

An additive mapping d:R →R is said to be a derivation of R if d(xy) = d(x)y + xd(y) for all x, y ∈ R. A derivation d is said to be inner if there exists a ∈R such that d(x) = ax - xa for all x∈R.

In [1], Bell and Kappe proved that if a derivation d of a prime ring R which acts as homomorphisms or antihomomorphisms on a nonzero right ideal of R then d = 0 on R.

The objective of the present paper is to extend the above mentioned result as follows:

Theorem1.1. Let *R* be a prime ring. Suppose *d:R∈R* is an additive mapping satisfying *d*(*xy*) = *d*(*x*)*y* + *xd*(*y*) for all *x, y* *R*.

1. If *d* acts as a 3-homomorphism on *R*, then *d* = 0.
2. (ii) If *d* acts as a 3-antihomomorphism on *R*, then *d* = 0 ≠ (In the case if d 0, then

R is commutative).

Before proving our main result, we need following lemma:

Lemma1.1 ([6], Lemma 1). *Let d be a derivation of a prime ring R and a be a non zero element of R. If ad(x) = 0 for all x ∈R, then either a = 0 or d = 0.*

Proof of Theorem 1.1. (i) If *d* acts as a 3-homomorphism of *R*, then we have

(1.1) *d*(*xyzt*) = *d*(*xyz*)*t* + *xyzd*(*t*) = *d*(*x*)*d*(*y*)*d*(*z*)*t* + *xyzd*(*t*) for all *t, x, y, z* ∈*R*.

On the other hand

(1.2) *d*(*xyzt*) = *d*(*x*)*d*(*y*)*d*(*zt*) = *d*(*x*)*d*(*y*)*d*(*z*)*t* + *d*(*x*)*d*(*y*)*zd*(*t*) for all *t, x, y, z* ∈*R*.

On comparing equations (1.1) and (1.2), we get *d*(*x*)*d*(*y*)*zd*(*t*) = *xyzd*(*t*). This implies that (*d*(*x*)*d*(*y*) - *xy*)*zd*(*t*) = 0. Hence using the primeness of *R*, we obtain either *d*(*x*)*d*(*y*) - *xy* = 0 for all *x, y*∈*R* or *d*(*t*) = 0 for all *t* ∈*R*. Assume that *d* ≠0. Then *d*(*x*)*d*(*y*) - *xy* = 0 for all *x, y*∈*R*. Replacing *y* by *yt* in the last expression, we get *d*(*x*) *d*(*y*)*t* + *d*(*x*)*yd*(*t*) - *xyt* = 0. Hence *d*(*x*)*yd*(*t*) = 0 for all *t, x, y* ∈*R*. Since *R* is prime, the above relation yields that *d*(*x*) = 0 for all *x* ∈*R*.

This leads to a contradiction.

(ii) Suppose *d* acts as the 3-antihomomorphism of *R* and assume that *d* ≠ 0. By the given assumption, we have

d(*xyz*) = *d*(*xy*)*z* + *xyd*(*z*) = *d*(*x*)*yz* + *xd*(*y*)*z* + *xyd*(*z*) = *d*(*z*)*d*(*y*)*d*(*x*) for all *x, y, z* ∈*R*. That is,

(1.3)*d*(*z*)*d*(*y*)*d*(*x*) = *d*(*x*)*yz* + *xd*(*y*)*z* + *xyd*(*z*) for all *x, y, z* ∈*R*.

Replacing *x* by *xz* in (1.3), we get

d(*z*)*d*(*y*)*d*(*x*)*z* + *d*(*z*)*d*(*y*)*xd*(*z*) = *d*(*x*)*zyz* + *xd*(*z*)*yz* + *xzd*(*y*)*z* + *xzyd*(*z*).

Since *d* acts as the 3-antihomomorphism on *R*, we get

d(*xyz*)*z* + *d*(*z*)*d*(*y*)*xd*(*z*) = *d*(*x*)*zyz* + *xd*(*z*)*yz* + *xzd*(*y*)*z* + *xzyd*(*z*) for all *x, y, z*∈*R*.

This implies that

d(*xy*)z^2 + *xyd*(*z*)*z* + *d*(*z*)*d*(*y*)*xd*(*z*) = *d*(*x*)*zyz* + *xd*(*z*)*yz* + *xzd*(*y*)*z* + *xzyd*(*z*)

for all *x, y, z*∈*R*. This further implies that

d(*xy*)z^2+*xd*(y^2)+*xyd*(*z*)*z*+*d*(*z*)*d*(*y*)*xd*(*z*)=*d*(*x*)*zyz*+*xd*(*z*)*yz*+*xzd*(*y*)*z*+*xzyd*(*z*).

Taking *y* = *z* in the last expression, we arrive at

(1.4) *d* z^2 *xd*(*z*) = *x*z^2(*z*) for all *x, z*∈*R*.

Replacing *x* by *tx* in (1.4), we get

(1.5) *d* z^2*txd*(*z*) = *txd*(*z*) for all *t, x, z*∈*R*.

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Left multiplying (1.4) by t, we obtain

(1.6) $td(z)^2 xd = txd(z)$ for all $t, x, \in R$.

On comparing (1.5) and (1.6), we arrive at $dz^2 txd(z)$- $td(z)^2(xd(z)$ = 0 for all $t, x, \in R$. That is, $[d(z)^2 xd(z) = 0$ for all $t, x, z \in R$. Suppose $D(t) = [d(z)^2$is a nonzero derivation of R, then in view of Lemma 1.1, we get $xd(z) = 0$ for all $x, z \in R$ and hence by the primeness of R, $d(z) = 0$ for all $z \in R$, a contradiction. Therefore, we must have $D(t) = [d(z)^2 = 0$ for all $t, z \in R$. That is, $d(z)^2 t = td(z)^2$ (for all $t, z \in R$. Thus $d(x) = d(xyy) = d(y)d(y)d(x) = d(y^2) = d(x)\, d(y^2) = d(x)d(y)d(y) = d(y^2 x)$ that is, $d([x,(y)^2\ ]) = 0$ for all $x, y \in R$. Replacing x by xy, we get

$0 = d([xy, y^2])$

$= d([x, y^2]y)$

$= [x,y^2\ ]d(y)$ for all $x, y \in R$.

Suppose $x \longmapsto [x,y^2\ ]$ is a nonzero derivation of R, then in view of Lemma 1.1, we get $d(y) = 0$ for all $y \in R$, a contradiction. Therefore we must have $[x,y^2\ ] = 0$ for all $x, y \in R$. This shows R is commutative. This completes the proof.

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