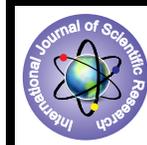


Mathematics and Music



Mathematics

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ABSTRACT

Music is a science which must have determined rules. These rules are drawn from a principle which should be evident and this principle cannot be known without the help of mathematics. Music theory as well as musical composition requires certain abstract way of thinking and contemplation which are very close to pure mathematical thought. Music and mathematics are both deeply expressive languages whose mysteries are revealed through pattern and serendipity

Leon Battista Alberti (1407-1472) says "We shall therefore borrow all our Rules for the Finishing our Proportions, from the Musicians, who are the greatest Masters of this Sort of Numbers, and from those Things wherein Nature shows herself most excellent and complete."

According to Pythagoras "There is geometry in the humming of the strings, there is music in the spacing of the spheres." Schools of Pythagoras, Plato and Aristotle considered music as part of Mathematics. In Greek, mathematics contains four sections: Music, Astronomy, Number Theory and Geometry. This division of mathematics is called quadrivium (the four ways) [1]. Several important mathematicians of the seventeenth and eighteenth centuries were also music theorists. The situation changed with the Renaissance, when theoretical music became an independent field, but strong links with mathematics were maintained [2]. Music makes use of a symbolic language together with a rich system of notation, including diagrams which are similar to mathematical graphs of discrete functions in two dimensional Cartesian coordinates, the x- coordinates representing time and the y-coordinates representing pitch. Music theorists used these Cartesian diagrams long before they were introduced in geometry. Besides abstract language and notation, mathematical notations like symmetry, periodicity, continuity, discreteness are omnipresent in music. Lengths of musical intervals, rhythm, tempi and several other mathematical notions are naturally expressed by numbers. The mathematical use of the word "harmonic" has its origin in music theory. The composer Milton Babbitt writes that a musical theory should be "stable as a connected set of axioms, definitions and theorems, the proofs of which are derived by means of an appropriate logic" [2]. It is important to believe that there are contributions in both directions. Mathematical language and mathematical ideas shaped the language and the concepts of music theory. Questions and problems arising in music theory give motivation for investigations in mathematics and physics. Milton Babbitt uses group theory and set theory in his theoretical musical teaching and in his compositions. Phenomena like the production of beats or the production of the harmonic frequencies were noticed and discussed by music theorists several decades before they were explained by mathematical and physical theories.

Pythagoras (sixth century B.C.) is regarded as the first music theorist. The major musical discovery of Pythagoras is the relation of musical intervals with ratios of integers. This is described by Jamblichus [3] in these terms: Pythagoras was "reasoning with himself, whether it would be possible to devise instrumental assistance to the hearing, which could be firm and unerring, such as the sight obtains through the compass and the rule". Pythagoras heard the different sounds produced by hammers beating an anvil. He realized that the pitch, that is, the musical note, which

was produce by a particular hammer, depends only on weight of the hammer and not on place where the hammer hit the anvil or on the magnitude of the stroke. Pythagoras realized also that the compass of a musical interval between two notes produced by two different hammers depends only on the relative weights of the hammers, and in particular that the consonant musical intervals, which in classical Greek music were the intervals of octave, of fifth and of fourth, correspond, in terms of weights, to the numerical fraction $2/1$, $3/2$ and $4/3$ respectively. Thus Pythagoras thought that the relative weights of two hammers producing an octave is $2/1$, and so on. As soon as this idea occurred to him, Pythagoras went home and performed several experiments using different kinds of instruments, which confirmed the relationship between musical intervals and numerical fractions. Some of these experiments consisted of listening to the pitch produced by the vibrations of string that have the same length; he had suspended the strings from one end and attached different weights to other end. Other experiments involved strings of different lengths, which he had stretched end to end, as in musical instruments. He also did experiments on pipes and other wind instruments, and all these experiments confirmed him in his idea that musical intervals correspond in an immutable way to definite ratios of integers, whether these are ratios of lengths of pipes, lengths of strings, weights, etc [4].

Theon of Smyrna treatise describes other experiments which illustrate this relation between musical intervals and quotients of integers. He relates, for instance, that the Pythagoreans considered a collection of vases, filled partially with different quantities of the same liquid, and observed on them the "rapidity and the slowness of the movements of air vibrations". By hitting these vases in pairs and listening to the harmonies produced, they were able to associate numbers to consonances. This result is again that the octaves, fifths and fourths correspond respectively to the fractions $2/1$, $3/2$ and $4/3$, in terms of the quotients of levels of the liquid. These experiments were repeated and reinterpreted by the acousticians of the seventeenth century. The ideas and observations of Pythagoras and his school established the relation between musical intervals and ratios of integers.

The arithmetic of musical intervals involves in a very natural way the theory of logarithms. Music theorists dealt intuitively with logarithms long before these were defined as an abstract mathematical notion. It was only in the seventeenth century that logarithms were formally introduced in music theory by Isaac Newton, and then by Leonhard Euler and Jacques Lambert.

It is interesting to go through the music theory part of a mathematical treatise of the classical Greek era. The title of part 2 of Theon's mathematical treatise is "A book con-

taining the numeric laws of music." In the introduction, he says, "Harmony is spread in the world, and offers itself to those who seek it only if it is revealed by numbers". The first part of this sentence, that "Harmony is spread in the world", has been repeated throughout the ages, and it was at the basis of a strong feeling of cosmic structure and order. The second part of Theon's sentence, that "harmony is revealed by numbers", has also been repeated throughout the ages.

There are several kinds of semitones used in ancient Greek music, two of which are the "diatonic semitone" and the "chromatic semitone", the values of which are $16/15$ and $25/24$ respectively. The reasons why ancient Greek music used semitones of $16/15$ or $25/24$ are certainly related to the fact that these intervals are acceptable by the ear. But it is also a fact that the ancient Greek musicologists linked to deal with super particular ratios derived from 2,3 and 5, that is, fractions of the form $(n+1)/n$ with numerator and denominator having only 2, 3 and 5 as prime factors. Some useful musical intervals are $2/1$ octave, $3/2$ fifth, $4/3$ fourth, $5/4$ major third, $6/5$ minor third, $9/8$ major tone, $10/9$ minor tone, $16/15$ diatonic semitone, $25/24$ chromatic semitone and $81/80$ comma of Didymus [4].

Scales are building blocks for musical compositions. The major part of post- Renaissance Western European classical music uses a very limited number of scales. The theory of harmony in classical Greece included a complicated and very subtle system of scales. Greek mathematical treatises usually contain a description of scales in terms of fractions, with a discussion of the logic behind the definitions. The scale of Pythagoras is defined by the following sequence of numbers:

1, $9/8$, $81/64$, $4/3$, $3/2$, $27/16$, $243/128$, 2.

Composer Rameau made a real synthesis between music as an art whose aim is to express and to create emotions, and music as a mathematical science with a deductive approach and rigorous rules. Rameau argued that since the fundamental objects of mathematics are derived from the sequence of positive integers and since this sequence is contained in music, then mathematics it self is a part of music. These reflections provoked a dispute. The conclusion is that the interaction between music and mathematics was always friendly.

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