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# On Ternary Cubic Diophantine Equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$

MATHEMATICS

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**ABSTRACT**

*The non-homogeneous cubic equation with three unknowns represented by the Diophantine equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting properties among the solutions are presented.*

## 1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-17] for cubic equations with three unknowns. This communication concerns with yet another interesting equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$ representing non-homogeneous cubic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

## 2. Notations used

$T_{m,n}$ = *Triangular number of rank n*

$P_m^{\,n}$ = *Pyramidal number of rank n*

$P(n)$ = *Pronic number of rank n*

$Star_n$ = *Star number of rank n*

$FN_n^{\,4}$ = *Four dimentional figurative number whose generating polygon is a squre*

$SO_n$ = *Stella octangula number of rank n*

$CP_n^{\,6}$ = *Centered hezagonal pyramidal number of rank n*

$Bq_n$ = *Biquadratic number of rank n*

## 3. Method of Analysis:-

The ternary cubic Diophantine equation to be solved for its non-zero distinct integral solution is $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$ (1)

The substitution of linear transformations $(u \neq v \neq o)$

$x = u + v,\ y = u - v$ (2)

In (1) leads to $(u + 1)^2 + 11v^2 = 15z^3$ (3)

Different patterns of solutions of (1) are presented below

**Pattern – I**

Write 15 as

$$15 = \left(2 + i\sqrt{11}\right)\left(2 - i\sqrt{11}\right) \tag{4}$$

Assume that $z = a^2 + 11b^2$ (5)

Use (4) and (5) in (3) and employing the method of factorization. Define

$$(u+1) + i\sqrt{11}v = (2 + i\sqrt{11})\ (a + i\sqrt{11}b)^3 \tag{6}$$

Equating the real and imaginary parts in (6) we get.

$$\left.\begin{aligned} u &= 2a^3 + 121b^3 - 33a^2b - 66ab^2 - 1 \\ v &= a^3 - 22b^3 + 6a^2b - 33ab^2 \end{aligned}\right\} \tag{7}$$

In view of (7), the solution of (1) can be written as

$$\left.\begin{aligned} x &= 3a^3 + 99b^3 - 27a^2 b - 99ab^2 - 1 \\ y &= a^3 + 143b^3 - 39a^2b - 33ab^2 - 1 \\ z &= a^2 + 11b^2 \end{aligned}\right\} \tag{8}$$

The equation (8) represents non-zero distinct integral solution of (1) on two parameters.

A few interesting properties observed are as follows.

(i) $3z\ (a,a) is\ a\ perfect\ square$

(ii) $2z\ (a,a) is\ a\ Nasty\ number$

(iii) $3[\ y(a,a) + 1]\ is\ a\ cubical\ integer$

(iv) $[x(a,-a) + 1] is\ a\ perfect\ square$

(v) $x(a,1) - 3y(a,1) - 90t_{4,a} + 330 = 0$

(vi) $x(a,1) + 30z(a,1) - 6P_a^{\,5} \equiv 32(Mod\ 99)$

(vii) $x(a,1) + y(a,1) - 8P_a^{\,5} + 140t_{3,a} \equiv 54(Mod\ 62)$

**Pattern – II**

Equation (3) can be written as

$$(u+1)^2 + 11v^2 = 15z^3 * 1 \tag{9}$$

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Assume that $z = a^2 + 11b^2$ (10)

$$\left.\begin{aligned} &\text{Write 1 as } 1 = \frac{\left(5 + i\sqrt{11}\right)\left(5 - i\sqrt{11}\right)}{36} \\ &\text{Also } 15 = \left(2 + i\sqrt{11}\right)\left(2 - i\sqrt{11}\right) \end{aligned}\right\} \quad (11)$$

Use (10) and (11) in (9) and employing the method of factorization. Define

$$(u+1) + i\sqrt{11}v = {}^{1}/_{6}\{(2 + i\sqrt{11})\,(5 + i\sqrt{11})\,(a + i\sqrt{11}b)^3\} \quad (12)$$

Equating the real and imaginary parts in (12) we get.

$$\left.\begin{aligned} u &= {}^{1}/_{6}\{-a^3 + 847b^3 + 33ab^2 - 231a^2b - 6\} \\ v &= {}^{1}/_{6}\{7a^3 + 11b^3 - 231ab^2 - 3a^2b\} \end{aligned}\right\} \quad (13)$$

Substituting (13) in (2), the corresponding integer solution of (1) are given by

$$\left.\begin{aligned} x &= a^3 + 143b^3 - 33ab^2 - 39a^2b - 1 \\ y &= {}^{1}/_{6}\{-8a^3 + 836b^3 + 264ab^2 - 228a^2b - 6\} \\ z &= a^2 + 11b^2 \end{aligned}\right\} \quad (14)$$

Our interest is to obtain the integer solutions, so that the values of x and y are integers for suitable choices of the parameters a and b.

*put* $a = 6A$, $b = 6B$

$$\left.\begin{aligned} x &= 216A^3 + 30888B^3 - 7128AB^2 - 8424A^2B - 1 \\ y &= -288A^3 + 30096B^3 + 9504AB^2 - 8208A^2B - 1 \\ z &= 36A^2 + 396B^2 \end{aligned}\right\} \quad (15)$$

The equation (15) represents non-zero distinct integral solution of (1) on two parameters.

A few interesting properties observed are as follows.

(i) $z(A, A) - 432t_{4,A} = 0$

(ii) $3z(A, A+1) = 1296pr_A \equiv 108 \pmod{1080}$

(iii) $z(A, 1) - 36t_{4,A} = 396$

(iv) $x\,(A,1) - 432{P_A}^5 + 17280t_{3\,,A} \equiv\ 647\ (mod\ 1512)$

(v) $y\,(A,1) + 576{P_A}^5 + 15840t_{3\,,A} \equiv\ 12671\ (mod\ 17424)$

(vi) $x\,(A,1) - 6z(A,1) + 16848t_{3\,,A} \equiv 1295(mod\ 1296)$

(vii) $y\,(A,1) + 8az(A,1) + 8208Pr_A\ \equiv 9215\ (mod\ 20880)$

**Pattern – III**

Again equation (3) can be written as

$$(u+1)^2 + 11v^2 = 15z^3 * 1 \qquad (16)$$

Assume that $z = a^2 + 11b^2$ (17)

$$\left.\begin{aligned} &\text{Write 15 as } 15 = \frac{(6+\ i3\sqrt{11})(6 - i3\sqrt{11})}{9} \\ &\text{Also}\qquad 1 = \frac{(5+\ i\sqrt{11})(5 - i\sqrt{11})}{36} \end{aligned}\right\} \qquad (18)$$

Use (18) and (17) in (16) and employing the method of factorization. Define

$$(u+1) + i\sqrt{11}\,v = {}^{1}/_{3}\,.\,{}^{1}/_{6}\{(6 + i3\sqrt{11})(5 + i\sqrt{11})(a + i\sqrt{11}b\ \ )^3\} \quad (19)$$

Equating the real and imaginary parts in (19) we get

$$\left.\begin{aligned} u &= {}^{1}/_{18}\{-3a^3 + 99ab^2 - 693a^2b + 2541b^3 - 18\} \\ v &= {}^{1}/_{18}\{21a^3 - 9a^2b - 693ab^2 + 33b^3\} \end{aligned}\right\} \qquad (20)$$

Substituting (20) in (2), the corresponding integer solution of (1) are given by

$$\left.\begin{aligned} x &= a^3 - 33ab^2 - 39a^2b + 143b^3 - 1 \\ y &= {}^{1}/_{18}\{-24a^3 + 792ab^2 - 684a^2b + 2508b^3 - 18\} \\ z &= a^2 + 11b^2 \end{aligned}\right\} \qquad (21)$$

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Our interest is to obtain the integer solutions, so that the values of x and y are integers for suitable choices of the parameters a and b.

Replace a by 18A and b by 18B we have

$$\left.\begin{aligned} x &= 5532A^3 - 192456AB^2 - 227448A^2B + 833976B^3 - 1 \\ y &= -7776A^3 + 256608AB^2 - 221616A^2B + 812592B^3 - 1 \\ z &= 324A^2 + 3564B^2 \end{aligned}\right\} \qquad (22)$$

The equation (22) represents non-zero distinct integral solution of (1) on two parameters.

A few interesting properties observed are as follows.

(i) $x\,(A,1) - 18Az\,(A,1) + 227448pr_A \equiv 17495\ (mod\ 29160)$

(ii) $z\,(A, A+1) - 3888t_{4,A} \equiv 3564\ (mod\ 7128)$

(iii) $x\,(A,1) + y(A,1) + 3888{P_A}^5 + 447120t_{4\,,A} \equiv 42766\ (mod\ 64152)$

(iv) $z(A,A) - 3888t_{4\,,A} = 0$

(v) $\{x(A,A)+1\} + [y(A,A)+1] is\ a\ cubical\ integer$

(vi) $z\,(A, A(A+1)) - 4272\ {FN_n}^4 - 1424{P_A}^5 - 3246_{4\,,A} = 0$

(vii) $z\,(A, A+1) - 1224720{p_A}^5 + 384912t_{3\,,A} - 1662120t_{4\,,A} \equiv 833975\ (mod\ 2501928)$

**<u>Pattern – IV</u>**

Again, Equation (3) can be written as

$$(u+1)^2 + 11v^2 = 15z^3 * 1 \qquad (23)$$

Assume that $z = a^2 + 11b^2$ (24)

$$\left.\begin{aligned} &\text{Write 15 as } 15 = \frac{(4+i2\sqrt{11})(4-i2\sqrt{11})}{4} \\ &\text{Also} \quad 1 = \frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100} \end{aligned}\right\} \qquad (25)$$

Use (25) and (24) in (23) and employing the method of factorization. Define

$$(u+1) + i\sqrt{11}\,v = {}^1\!/_2 . {}^1\!/_{10} \{(4+i2\sqrt{11})(1+i3\sqrt{11})(a+i\sqrt{11}b)^3\} \qquad (26)$$

Equating the real and imaginary parts in (26) we get.

$$u = 1/_{20}\ \{-62a^3 + 1694b^3 + 2046ab^2 - 462a^2b - 20\} \quad (27)$$

$$v = 1/_{20}\ \{682b^3 + 14a^3 - 186a^2b - 462ab^2\} \quad (28)$$

Substituting (27) and (28) in (2), the corresponding integer solution of (1) are given by

$$\left.\begin{aligned} x &= 1/_{20}\ \{-48a^3 + 2376b^3 + 1584ab^2 - 648a^2b - 20\} \\ y &= 1/_{20}\ \{-76a^3 + 1012b^3 + 2508ab^2 - 276a^2b - 20\} \\ z &= a^2 + 11b^2 \end{aligned}\right\} \quad (29)$$

Our interest is to obtain the integer solutions, so that the values of x and y are integers for suitable choices of the parameters a and b.

put $a$ =20 $A$ and $b$ =20 $B$

$$\left.\begin{aligned} x &= -\ 19200A^3 + 950400B^3 + 633600AB^2 - 259200A^2B - 1\} \\ y &= -30400A^3 + 404800B^3 + 1003200AB^2 - 110400A^2B - 1\} \\ z &= 400A^2 + 4400B^2 \end{aligned}\right\} \quad (30)$$

The equation (30) represents non-zero distinct integral solution of (1) on two parameters.

A few interesting properties observed are as follows.

(i) $z\ (A, A+1) - 4800t_{4,A} \equiv\ 0\ (mod\ 8800)$

(ii) $x\ (A,1) + 48z(A,1) + 259200pr_A \equiv\ 268799\ (mod\ 892800)$

(iii) $y\ (A,1) + 76z(A,1) + 110400t_{4,A} \equiv\ 739199\ (mod\ 1003200)$

(iv) $x\ (A,1) + z(A,1) - 19200{CP_A}^6 + 258800t_{4,A} \equiv\ 323899\ (mod\ 633600)$

(v) $z\ (A, A(A+1)) - 52800{FN_A}^4 + 8800{CP_A}^6 + 9200t_{4,a} = 0$

(vi) $x\ (A, A+1) - 652800So_A - Star_A + 4118394t_{4,A} \equiv\ 691198\ (mod\ 4137594)$

(vii) $z\ (A^2, A^2) - 4800Bq_A = 0$

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## 4. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary cubic Diophantine equation represented by

$$3(x^2 + y^2) - 5xy \quad + x + y + 1 = 15z^3$$

One can also search for other patterns of solutions for the above equation.

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