ORIGINAL RESEARCH PAPER
Volume-6 | Issue-12 | December-2017 | ISSN No 2277 - 8179 | IF : 4.176 | IC Value : 93.98
INTERNATIONAL JOURNAL OF SCIENTIFIC RESEARCH

ON INVESTIGATION OF ASYMPTOTIC PROPERTIES OF ESTIMATORS USED IN ESTIMATION OF EXPECTED SHORTFALL OF A FINANCIAL PORTFOLIO BASED ON EXPONENTIAL SMOOTHING TECHNIQUE

Statistics

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ABSTRACT

Investigation of asymptotic properties of estimator(s) is important in any estimation procedure. This paper adopts a model according to Mwita (2003) to model Expected Shortfall of a financial portfolio. We propose, based on exponential smoothing technique, the estimators of the conditional mean, conditional volatility and Value-at-Risk. We then proceed to investigate their consistency under some specified conditions.

KEYWORDS

Value- at- Risk (VaR), Expected Shortfall (ES), Exponential Smoothing, Conditional mean, Conditional Volatility.

1. INTRODUCTION

Exponential Smoothing is a popular technique used to produce a smoothed time series. Unlike the single moving averages whereby the past observations are weighted equally, Exponential Smoothing assigns exponentially decreasing weights as the observation get older. In other words, recent observations are given relatively more weight in forecasting than the older observations.

In the case of moving averages, the weights assigned to the observations are the same and are equal to 1/N. In exponential smoothing, however, there are one or more smoothing parameters to be determined (or estimated) and these choices determine the weights assigned to the observations.

Our underlying process of interest is of the form $X_t = \mu_t + \sigma_t e_t$, see Mwita (2003), where μ_t is the conditional mean function of X_t given the past information, F_{t-1}, σ_t is the conditional volatility function of X_t given the past information, F_{t-1} and e_t are the error terms, which we assume to be independent of F_{t-1} and are normally distributed with mean 0 and variance 1.

In section two, we propose the estimators for the conditional mean and conditional volatility, and derive their Consistency under some conditions. Section three gives the estimators for Value-at-Risk and Expected Shortfall and a proof that Value-at-Risk is consistent. In section four, we give empirical results and their interpretation. Finally, we give conclusion and our recommendations in section five.

2. Estimators for the conditional mean and conditional volatility

Let $X_1,...,X_H,X_{H+1},...,X_n$ be a sequence of returns, where H is the size of the horizon we take at time t and n is the sample size large enough such that we can have a number of horizons that can give a vector of estimates. The estimator for the conditional mean is given by

$$\hat{\mu}_t = \sum_{j=t-H+1}^{t} \omega_J X_j \tag{2.1}$$

where

$\omega_j = (1-\lambda)\lambda^{t-j}$ are exponentially decreasing weights

$\lambda \in (0,1)$, is the smoothing constant

t is the current time period

H is the time horizon.

We define the estimator for conditional volatility under the following situations:

1. When the conditional mean, μ_t, is known, the estimator is given by

$$\hat{\sigma}_t^2 = \sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}X_j^2 \tag{2.2}$$

2. When the conditional mean, μ_t , is unknown, the estimator is given by

$$\hat{\sigma}_t^2 = \sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}\left(X_j - \hat{\mu}_t\right)^2 \tag{2.3}$$

Assumptions

(A1) $E(e_i) = 0$

(A2) $\sigma^2 = E(e_i^2) < \infty$

(A3) $\underset{|i-j|\to\infty}{Lim}\ Cov(X_i, X_j) = 0$ where i represents an observation in one horizon e.g $X_1, X_2, ..., X_H$ and j represents an observation in another horizon e.g $X_2, X_2, ..., X_{H+1}$

Theorem 2.1 (Consistency for the conditional mean estimator)
Let $X_1, X_2, ..., X_H, X_{H+1}, ..., X_n$ be a random sample of returns with mean μ_t and variance σ_t^2 .Then, $\hat{\mu}_t = \sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}X_j$ is a weakly consistent estimator for μ_t .

Proof

We show that $\underset{H\to\infty}{Lim} E(\hat{\mu}_t) = \mu_t$ and $\underset{H\to\infty}{Lim} Var(\hat{\mu}_t) = 0$. We proceed as follows:

$$\hat{\mu}_t = \sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}X_j$$

$$= (1-\lambda)\lambda^t \sum_{j=t-H+1}^{t}\frac{1}{\lambda^j}X_j$$

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$$E(\hat{\mu}_t) = (1-\lambda)\lambda^t \sum_{j=t-H+1}^{t} E\left(\frac{X_j}{\lambda^j}\right) \tag{2.4}$$

$$= (1-\lambda)\lambda^t \sum_{j=t-H+1}^{t}\frac{\mu_t}{\lambda^j} \tag{2.5}$$

$$= (1-\lambda)\lambda^t H \frac{\mu_t}{\lambda^j} \tag{2.6}$$

$$= (1-\lambda)\lambda^{t-j} H\mu_t$$

$$= (1-\lambda)\lambda^{H-1} H\mu_t \tag{2.7}$$

$$\underset{H\to\infty}{Lim} E(\hat{\mu}_t) = \mu_t \tag{2.8}$$

$$Var(\hat{\mu}_t) = (1-\lambda)^2\lambda^{2t} \sum_{j=t-H+1}^{t} \mathrm{var}\left(\frac{X_j}{\lambda^j}\right) - Cov\left(\frac{X_i}{\lambda^i}, \frac{X_j}{\lambda^j}\right) \tag{2.9}$$

$$= (1-\lambda)^2\lambda^{2t} \sum_{j=t-H+1}^{t}\left(\frac{\sigma_t^2}{\lambda^{2j}}\right) \tag{2.10}$$

$$= (1-\lambda)^2\lambda^{2t} H\left(\frac{\sigma_t^2}{\lambda^{2j}}\right) \tag{2.11}$$

$$= (1-\lambda)^2\lambda^{2(t-j)} H\sigma^2$$

t

$$=(1-\lambda)^2\lambda^{2(H-1)}H\sigma_t^2 \quad (2.12)$$

$$\lim_{H\to\infty}(\text{var}\,\hat{\mu}_t) = \lim_{H\to\infty}\left\{(1-\lambda)^2\lambda^{2(H-1)}H\sigma_t^2\right\}=0$$

which implies that $\hat{\mu}_t \xrightarrow{p} \mu_t$.

Remarks

a) The time horizon selected is large ($H>30$). Therefore, by the central limit theorem, the returns are normally distributed with mean μ_t and variance σ_t^2 i.e. $X_j \sim N(\mu_t,\sigma_t^2)$.It is in this respect that (2.4) leads to (2.5).

b) Equation (2.6) is obtained by summing equation (2.5) over the horizon, H .Similarly, equation (2.11) is obtained from equation (2.10).

c) Equations (2.7) and (2.12) are obtained by noting that $j=t-H+1$.

d) Equation (2.8) is obtained from equation (2.7) by noting that the weights sum up to 1 as $H\to\infty$.

e) By assumption *(A3)*, (2.10) is obtained from equation (2.9).

Theorem 2.2(Consistency for $\hat{\sigma}_t$ when $\hat{\mu}_t$ is known)

Let $X_1,...,X_H,X_{H+1},...X_n$ be a sequence of returns with mean μ_t and variance , σ_t^2 . Then, $\hat{\sigma}_t=\left[\sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}X_j^2\right]^{1/2}$ is a simple consistent estimator for σ_t.

Proof of theorem 2.2

We show that $\lim_{H\to\infty}E(\hat{\sigma}_t^2)=\sigma_t^2$ and $\lim_{H\to\infty}Var(\hat{\sigma}_t^2)=0$. We proceed as follows:

$$\hat{\sigma}_t^2=\sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}X_j^2$$

$$=(1-\lambda)\lambda^t\sum_{j=t-H+1}^{t}\frac{1}{\lambda^j}X_j^2$$

$$E(\hat{\sigma}_t^2)=(1-\lambda)\lambda^t\sum_{j=t-H+1}^{t}E\left(\frac{X_j^2}{\lambda^j}\right) \quad (2.13)$$

$$=(1-\lambda)\lambda^t\sum_{j=t-H+1}^{t}\frac{\sigma_t^2}{\lambda^j} \quad (2.14)$$

$$=(1-\lambda)\lambda^t H\frac{\sigma_t^2}{\lambda^j} \quad (2.15)$$

$$=(1-\lambda)\lambda^{t-j}H\sigma_t^2$$

$$=(1-\lambda)\lambda^{H-1}H\sigma_t^2 \quad (2.16)$$

$$\lim_{H\to\infty}E(\hat{\sigma}_t^2)=\sigma_t^2 \quad (2.17)$$

2 2 2

$$Var(\hat{\sigma}_t^2)=(1-\lambda)^2\lambda^{2t}\sum_{j=t-H+1}^{t}\text{var}\left(\frac{X_j}{\lambda^j}\right)-Cov\left(\frac{X_i}{\lambda^i},\frac{X_j}{\lambda^j}\right) \quad (2.18)$$

$$=(1-\lambda)^2\lambda^{2t}\sum_{j=t-H+1}^{t}\left(\frac{\sigma_t^2}{\lambda^{2j}}\right) \tag{2.19}$$

$$=(1-\lambda)^2\lambda^{2t}H\left(\frac{\sigma_t^2}{\lambda^{2j}}\right) \tag{2.20}$$

$$=(1-\lambda)^2\lambda^{2(t-j)}H\sigma_t^2$$

$$=(1-\lambda)^2\lambda^{2(H-1)}H\sigma_t^2 \tag{2.21}$$

$$\underset{H\to\infty}{Lim}\left(\operatorname{var}\hat{\sigma}_t^2\right)=\underset{H\to\infty}{Lim}\left\{(1-\lambda)^2\lambda^{2(H-1)}H\sigma_t^2\right\}=0$$

which implies that $\hat{\sigma}_t \xrightarrow{p} \sigma_t$.

Remarks

a) Equation (2.14) is obtained from equation (2.13) by noting that $H>30$. That is, the time horizon we have selected is large. Therefore, by the central limit theorem, the returns are normally distributed with mean μ_t and variance σ_t^2 i.e. $X_j \sim N(\mu_t,\sigma_t^2)$. Hence, we have $E\left(\frac{X_j}{\lambda^j}\right)=\frac{\mu_t}{\lambda^j}$ and $Var\left(\frac{X_j}{\lambda^j}\right)=\frac{\sigma_t^2}{\lambda^{2j}}$. In a similar way, equation (2.19) is obtained from equation (2.20),under assumption *(A3).*

b) Equation (2.15) is obtained by summing equation (2.14) over the horizon, H. Similarly, equation (2.20) is obtained from equation (2.19).

c) Equations (2.16) and (2.21) are obtained by noting that $j=t-H+1$.

d) Equation (2.17) is obtained from equation (2.16) by noting that the weights sum up to 1 as $H\to\infty$.

Theorem 2.3 (Consistency for $\hat{\sigma}_t$ when $\hat{\mu}_t$ is unknown)

Let $X_1,\ldots,X_H,X_{H+1},\ldots X_n$ be a sequence of returns with mean μ_t and variance, σ_t^2. Then, $\hat{\sigma}_t=\left[\sum_{j=t-H+1}^{t}(1-\lambda)\lambda^{t-j}(X-\hat{\mu}_t)^2\right]^{1/2}$ is a simple consistent estimator for σ_t.

Proof of theorem 2.3

We show that $\underset{H\to\infty}{Lim}E(\hat{\sigma}_t^2)=\sigma_t^2$ and $\underset{H\to\infty}{Lim}Var(\hat{\sigma}_t^2)=0$. We proceed as follows:

$$\hat{\sigma}_t^2=\sum_{j=t-H+1}^{t}(1-\lambda)^{(t-j)}\left(X_j-\hat{\mu}_t\right)^2$$

$$=\sum_{j=t-H+1}^{t}(1-\lambda)^{(H-1)}\left(X_j-\hat{\mu}_t\right)^2$$

$$=(1-\lambda)^{(H-1)}\sum_{j=t-H+1}^{t}\left(X_j-\hat{\mu}_t\right)^2$$

$$\frac{\hat{\sigma}_t^2}{(1-\lambda)^{(H-1)}}=\sum_{j=t-H+1}^{t}\left(X_j-\hat{\mu}_t\right)^2$$

$$\frac{\hat{\sigma}_t^2}{(1-\lambda)^{(H-1)}\sigma_t^2}=\frac{\sum_{j=t-H+1}^{t}\left(X_j-\hat{\mu}_t\right)^2}{\sigma_t^2}$$ has a Chi-square distribution with $(H-1)$ degrees of freedom.

Therefore,

$$E\left[\frac{\hat{\sigma}_t^2}{(1-\lambda)^{(H-1)}\sigma_t^2}\right]=(H-1)$$

and

$$Var\left[\frac{\hat{\sigma}_t^2}{(1-\lambda)^{(H-1)}\sigma_t^2}\right]=2(H-1)$$

Hence,

$$E(\hat{\sigma}_t^2)=(1-\lambda)^{H-1}(H-1)\sigma_t^2$$

$$\lim_{H\to\infty} E(\hat{\sigma}_t^2)=\sigma_t^2$$

$$Var(\hat{\sigma}_t^2)=(1-\lambda)^{2(H-1)}2(H-1)\sigma_t^4$$

$$\lim_{H\to\infty} \mathrm{var}(\hat{\sigma}_t^2)=0$$

which implies that $\hat{\sigma}_t^2 \xrightarrow{p} \sigma_t^2$ and so is $\hat{\sigma}_t$. Hence, proved.

3. ESTIMATORS FOR VALUE AT RISK AND EXPECTED SHORTFALL

3.1 Estimator for Value at Risk (VaR)

The Value-at-Risk summarizes the expected maximum loss (or worst loss) over a target horizon at a given confidence level θ. In our case, we use a target horizon of 250 trading days and a confidence level of 99%.This means that the probability to lose at the next day more than the calculated Value-at-Risk is less than 1%.

We obtain today's returns, X_t, as $X_t=-Log\left(\frac{P_t}{P_{t-1}}\right)$ where P_t and P_{t-1} are today's and yesterday's share prices respectively.

Refer to the econometric model in section 1. The conditional θ-quantile function for this model is given by

$$VaR_{t,\theta}=\mu_t+\sigma_t e_\theta$$

The estimator for the conditional θ-quantile is given by

$$V\hat{a}R_{t,\theta}=\hat{\mu}_t+\hat{\sigma}_t e_\theta$$

where $\hat{\mu}_t$ and $\hat{\sigma}_t$ are as defined in equations (2.1) and (2.3) respectively.

3.1.1 Consistency for the estimator for Value at Risk

We show that $\lim_{H\to\infty} E(V\hat{a}R_{t,\theta})=VaR_{t,\theta}$ and $\lim_{H\to\infty} Var(V\hat{a}R_{t,\theta})=0$. We proceed as follows:

First, we check that;

$$E(V\hat{a}R_{t,\theta})=E(\hat{\mu}_t)+e_\theta\, E(\hat{\sigma}_t) \quad (3.1)$$

$$=(1-\lambda)^{H-1}H\mu_t+e_\theta(1-\lambda)^{H-1}(H-1)\sigma_t^2 \quad (3.2)$$

$$\lim_{H\to\infty} E(V\hat{a}R_{t,\theta})=\mu_t+\sigma_t e_\theta \quad (3.3)$$

$$=VaR_{t,\theta}$$

Next, we check the asymptotic variance.

$$Var(V\hat{a}R_{t,\theta})=Var(\hat{\mu}_t)+e_\theta^2\, Var(\hat{\sigma}_t)-2e_\theta Cov(\hat{\mu}_t,\hat{\sigma}_t)$$

$$=\left((1-\lambda)^2\lambda^{H-1}H\sigma_t^2\right)+e_\theta^2\left((1-\lambda)^{2(H-1)}2(H-1)\sigma_t^4\right) \quad (3.4)$$

$$=(1-\lambda)^2\lambda^{H-1}H\sigma_t^2\left(1+e_\theta^2(1-\lambda)^{(H-1)}\sigma_t^2\right)$$

$$\lim_{H\to\infty} Var(V\hat{a}R_{t,\theta})=0$$

Hence, proved.

Remarks

a) Equation (3.2) is obtained from equation (3.1) by using assumption *(A3)* and noting that $E(\hat{\mu}_t)=(1-\lambda)^{H-1}H\mu_t$ and $E(\hat{\sigma}_t)=(1-\lambda)^{H-1}(H-1)\sigma_t$.

b) Equation (3.3) is obtained from equation (3.2) by noting that the weights sum up to 1 as $H\to\infty$.

c) In equation (3.4), $Var(\hat{\mu}_t)$ and $Var(\hat{\sigma}_t)$ are obtained as in equations (2.12) and (2.21) respectively.

3.2 Estimator for the Expected Shortfall

Expected shortfall is the conditional expectation of loss given that the loss is beyond the VaR level, and measures how much one can lose on average in the states beyond the VaR level. Suppose X_t is a random variable denoting the negative returns of a given portfolio and $VaR_{t,\theta}$ is the VaR at the θ- confidence level. The Expected Shortfall is given by

$$ES_\theta = E[X_t \mid X_t \geq VaR_{t,\theta}]$$

The function for expected shortfall based on our econometric model is given by

$$\begin{aligned} ES_\theta &= E[\mu_t+\sigma_t e_t \mid \mu_t+\sigma_t e_t \geq \mu_t+\sigma_t e_\theta] \\ &= \mu_t+\sigma_t E[e_t \geq e_\theta] \end{aligned}$$

and the estimator for the expected shortfall is given by

$$\begin{aligned} E\hat{S}_\theta &= \hat{\mu}_t+\hat{\sigma}_t E[\hat{e}_t \geq \hat{e}_\theta] \\ &= \hat{\mu}_t+\hat{\sigma}_t \frac{1}{k}\sum_{i=1}^{k}\hat{e}_t I_{\{\hat{e}_t\geq\hat{e}_\theta\}} \end{aligned}$$

where $\hat{\mu}_t$ and $\hat{\sigma}_t$ are as defined in equations (2.1) and (2.3) respectively. k is the number of negative returns that exceed $VaR_{t,\theta}$ and $I_{\{\hat{e}_t\geq\hat{e}_\theta\}}$ is an indicator function defined as

$$I_{\{e_t\geq e_\theta\}} = \begin{cases} 1 \text{ if } \hat{e}_t \geq \hat{e}_\theta \\ 0 \text{ if } \hat{e}_t < \hat{e}_\theta \end{cases}$$

4. CONCLUSION AND RECOMMENDATIONS

4.1 Conclusion

In our study, we adopted a regression model according to Mwita (2003). In estimating the conditional mean and conditional volatility, we explored the exponential smoothing technique, whereby we assigned exponentially decreasing weights to the observations. We proved that the estimators for the conditional mean and conditional volatility are consistent. Further, we have given the estimators for the Value-at-Risk and Expected Shortfall and have shown that the estimator for Value-at-Risk is consistent.

4.2 Recommendations

Though the exponential smoothing technique has a number of merits as we have seen, we also note the following shortcomings:

a) Exponential smoothing lags. In other words, the forecast will be behind as the trend increases over time.

b) Exponential smoothing may fail to account for the dynamic changes at work in the real world, and the forecast may constantly require updating to respond to new information.

In this respect, other techniques, like general autoregressive conditional heteroscedastic (GARCH) estimation, may be explored, see Thorsten et al (1999/2000).

We also suggest that the following work may be done in future as a continuation of this research:

a) The choice of the optimal smoothing parameter.
b) The proof of the asymptotic normality for the estimators.

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