The main aim of this paper is to introduce new type of open sets namely $\beta_T$ open sets in tri topological spaces along with their several properties and characterization. As application to $\beta_T$ open set we introduce $\beta_T$ continuous functions and obtain some of their basic properties.

**KEYWORDS:**

$\beta_T$ open sets and $\beta_T$ continuous functions.
Definition 2.3: Let \((X, T, T_1, T_2)\) be a tri topological space. Let \(A \subset X\). An element \(x \in A\) is called \(\beta_T\) interior point of \(A\), if there exist a \(\beta_T\) open set \(V\) such that \(x \in V \subset A\). The set of all \(\beta_T\) interior points of \(A\) is called the \(\beta_T\) interior of \(A\) and is denoted by \(sp \text{ int}(A)\).

Theorem 2.4: Let \(A \subset X\) be a tri topological space. \(sp \text{ int}(A)\) is equal to the union of all \(\beta_T\) open sets contained in \(A\).

Note 2.5: \(sp \text{ int}(A)\) is \(\beta_T\) open sets.

2. \(sp \text{ int}(A)\) is \(\beta_T\) open sets.

Theorem 2.5: \(sp \text{ int}(A)\) is the largest \(\beta_T\) open sets contained in \(A\).

Theorem 2.6: \(A\) is \(\beta_T\) open if and only if \(A = sp \text{ int}(A)\).

Theorem 2.7: \(sp \text{ int}(A) \cup B = sp \text{ int}(A \cup B) \cup \text{ spcl}(B)\).

Definition 2.8: Let \((X, T, T_1, T_2)\) be a tri topological space. Let \(A \subset X\). The intersection of all \(\beta_T\) semi closed set containing \(A\) is called a \(\beta_T\) closure of \(A\) and is denoted as \(\text{spcl}(A)\).

Note 2.9: Since intersection of \(\beta_T\) closed sets is \(\beta_T\) closed set, \(\text{spcl}(A)\) is a \(\beta_T\) closed set.

Note 2.10: \(\text{spcl}(A)\) is the smallest \(\beta_T\) closed set containing \(A\).

Theorem 2.11: \(A\) is \(\beta_T\) closed set if and only if \(A = \text{spcl}(A)\).

Theorem 2.12: Let \(A\) and \(B\) be subsets of \((X, T, T_1, T_2)\) and \(x \in X\).

a) If \(A \subset B\), then \(\text{spcl}(A) \subset \text{spcl}(B)\).

b) \(x \notin \text{spcl}(A)\) if and only if \(A \cap U \neq \emptyset\) for every \(\beta_T\) open set \(U\) containing \(x\).

Theorem 2.13: Let \(A\) be a subset of \((X, T, T_1, T_2)\), if there exist an \(\beta_T\) open set \(U\) such that \(A \subset U \subset \text{spcl}(A)\), then \(A\) is \(\beta_T\) open.

Theorem 2.14: In a tri topological space \((X, T, T_1, T_2)\), the union of any two \(\beta_T\) open sets is always an \(\beta_T\) open set.

Proof: Let \(A\) and \(B\) be any two \(\beta_T\) open sets in \(X\).

Now \(A \cup B \subset \text{spcl}(\text{spint}(A)) \cup \text{spcl}(\text{spint}(B))\).

\[
\Rightarrow A \cup B \subset \text{spcl}(\text{spint}(A \cup B)).
\]

Hence \(A \cup B \subset \text{spcl}(B)\) is \(\beta_T\) open sets.

Theorem 2.17: Let \(A\) and \(B\) be subsets of \(X\) such that \(B \subset A \subset \text{spcl}(B)\), if \(B\) is \(\beta_T\) open set then \(A\) is also \(\beta_T\) open set.

3. \(\beta_T\) Continuity in Tri topological space

Definition 3.1: A function \(f\) from a tri topological space \((X, T, T_1, T_2)\) into another tri topological space \((Y, W, W_1, W_2)\) is called \(\beta_T\) continuous if \(f^{-1}(V)\) is \(\beta_T\) open set in \(X\) for each tri open set \(V\) in \(Y\).

Example 3.2: Let \(X = \{a, b, c\}, T_1 = \{\emptyset, \{a\}\}, T_2 = \{\emptyset, \{a, b\}\}, T_3 = \{\emptyset, \{a, b, c\}\}\).

Open sets in tri topological spaces are union of all three topologies.

Then tri open sets of \(X\) are \(\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}\).

Set \(\text{SP}(X)\) sets of \(X\) are \(\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}\).

Let \(Y = \{1, 2, 3\}\), \(W_1 = \{\emptyset, \{1\}\}\), \(W_2 = \{\emptyset, \{2\}\}\), \(W_3 = \{\emptyset, \{1, 2\}\}\).

The tri open sets of \(Y\) are \(\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}\).

Consider the function \(f: X \rightarrow Y\) is defined as

\[
f^{-1}(\{1\}) = \{\emptyset, \{a\}\}, \quad f^{-1}(\{2\}) = \{\emptyset, \{a, b\}\}, \quad f^{-1}(\{3\}) = \{\emptyset, \{a, b, c\}\}.
\]

Since the inverse image of each tri open set in \(Y\) under \(f\) is \(\beta_T\) open set in \(X\). Hence \(f\) is \(\beta_T\) continuous function.

Theorem 3.3: Let \((X, T_1, T_2, T_3)\) and \((Y, W_1, W_2, W_3)\) be two tri topological spaces. A function \(f: X \rightarrow Y\) is \(\beta_T\) continuous if and only if the inverse image of every \(\beta_T\) open set in \(Y\) is tri open in \(X\).

Proof: (Necessary): Let \(f: (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)\) be \(\beta_T\) continuous function and \(U\) be any \(\beta_T\) open set in \(Y\). Then \(Y - U\) is \(\beta_T\) closed in \(Y\). Since \(f\) is \(\beta_T\) continuous function, \(f^{-1}(Y - U) = X - f^{-1}(U)\) is \(\beta_T\) closed in \(X\). Hence \(f^{-1}(U)\) is \(\beta_T\) open in \(X\).
(Sufficiency): Assume that \( f^{-1}(V) \) is \( \beta_T \) open in \( X \) for each tri open set \( V \) in \( Y \). Let \( V \) be a closed set in \( Y \). Then \( Y - V \) is \( \beta_T \) open in \( Y \). By assumption \( f^{-1}(Y - V) = X - f^{-1}(V) \) is \( \beta_T \) open in \( X \) which implies that \( f^{-1}(V) \) is \( \beta_T \) closed in \( (X, T_1, T_2, T_3) \). Hence \( f \) is \( \beta_T \) continuous function.

**Theorem 3.4**: Let \( (X, T_1, T_2, T_3) \) and \( (Y, W_1, W_2, W_3) \) be two tri topological spaces. A function \( f : X \to Y \) is \( \beta_T \) continuous function if and only if \( f^{-1}(U) \) is \( \beta_T \) closed set in \( X \) whenever \( V \) is tri closed in \( Y \).

**Theorem 3.5**: Let \( f : (X, T_1, T_2, T_3) \to (Y, W_1, W_2, W_3) \) be a \( \beta_T \) continuous open function. If \( V \) is an \( \beta_T \) open set of \( Y \), then \( f^{-1}(V) \) is \( \beta_T \) open in \( X \).

**Proof**: First, let \( V \) be a \( \beta_T \) open set of \( Y \). There exist an \( \beta_T \) set \( W \) in \( Y \) such that \( V \subseteq W \subseteq spcl(V) \). Since \( f \) is tri open set, we have \( f^{-1}(V) \subseteq f^{-1}(W) \subseteq f^{-1}(spcl(V)) \subseteq spcl(f^{-1}(V)) \). Since \( f \) is \( \beta_T \) continuous, \( f^{-1}(W) \) is \( \beta_T \) open set in \( X \). By theorem 2.13, \( f^{-1}(V) \) is \( \beta_T \) open set in \( X \). The proof of the second part is shown by using the fact of first part.

**Theorem 3.6**: The following are equivalent for a function \( f : (X, T_1, T_2, T_3) \to (Y, W_1, W_2, W_3) \):

a) \( f \) is \( \beta_T \) continuous function;

b) The inverse image of each \( \beta_T \) closed set of \( Y \) is \( \beta_T \) closed in \( X \);

c) For each \( x \in X \) and each tri open set \( V \) in \( W \), there exist an \( \beta_T \) open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \);

d) \( spcl(f^{-1}(B)) \subseteq f^{-1}(spcl(B)) \) for every subset \( B \) of \( Y \);

e) \( f(spcl(A)) \subseteq spcl(f(A)) \) for every subset \( A \) of \( X \).

**Theorem 3.7**: If \( f : (X, T_1, T_2, T_3) \to (Y, W_1, W_2, W_3) \) and \( g : (Y, W_1, W_2, W_3) \to (Z, \eta_1, \eta_2, \eta_3) \) be two \( \beta_T \) continuous function then \( gof : (X, P_1, P_2, P_3) \to (Z, \eta_1, \eta_2, \eta_3) \) may not be \( \beta_T \) continuous function.

**Theorem 3.8**: Let \( f^{-1} : (X, T_1, T_2, T_3) \to (Y, W_1, W_2, W_3) \) be bijective. Then the following conditions are equivalent:

i) \( f \) is a tri \( \beta_T \) open continuous function.

ii) \( f \) is \( \beta_T \) closed continuous function.

iii) \( f^{-1} \) is \( \beta_T \) continuous function.

**Proof**: (i)\(\Rightarrow\) (ii) Suppose \( B \) is a tri closed set in \( X \). Then \( X - B \) is an \( \beta_T \) open set in \( Y \). Now by (i), \( f(X - B) \) is a \( \beta_T \) open set in \( Y \). Now since \( f^{-1} \) is bijective so \( f(X - B) = f^{-1}(B) \). Hence \( f(B) \) is a \( \beta_T \) closed set in \( Y \). Therefore \( f \) is a \( \beta_T \) closed continuous function.

(ii)\(\Rightarrow\) (iii) Let \( f \) be an \( \beta_T \) closed map and \( B \) be a tri closed set of \( X \). Since \( f^{-1} \) is bijective so \( (f^{-1})^{-1}(B) \) which is an \( \beta_T \) closed set in \( Y \). Hence \( f^{-1} \) is \( \beta_T \) continuous function.

(iii)\(\Rightarrow\) (i) Let \( A \) be a tri open set in \( X \). Since \( f^{-1} \) is a \( \beta_T \) continuous function so \( (f^{-1})^{-1}(A) = f(A) \) is a \( \beta_T \) open set in \( Y \). Hence \( f \) is \( \beta_T \) open continuous function.

**Theorem 3.9**: Let \( X \) and \( Y \) are two tri topological spaces. Then \( f : (X, P_1, P_2, P_3) \to (Y, W_1, W_2, W_3) \) is \( \beta_T \) continuous function if one of the following holds:

i) \( f^{-1}(spint(B)) \subseteq spint(f^{-1}(B)) \) for every tri open set \( B \) in \( Y \).

ii) \( spcl(f^{-1}(B)) \subseteq f^{-1}(spcl(B)) \) for every tri open set \( B \) in \( Y \).

**Proof**: Let \( B \) be any tri open set in \( Y \) and if condition (i) is satisfied then \( f^{-1}(spint(B)) \subseteq spint(f^{-1}(B)) \). We get \( f^{-1}(B) \subseteq spint(f^{-1}(B)) \) Therefore \( f^{-1}(B) \) is a \( \beta_T \) open set in \( X \). Hence \( f \) is \( \beta_T \) continuous function. Similarly, we can prove (ii).

**Theorem 3.10**: A function \( f : (X, P_1, P_2, P_3) \to (Y, W_1, W_2, W_3) \) is called \( \beta_T \) open continuous function if and only if \( f(spint(A)) \subseteq spint(f(A)) \) for every tri open set \( A \) in \( X \).

**Proof**: Suppose that \( f \) is a \( \beta_T \) open continuous function.

Since \( sp \int(A) \subseteq A \) so \( f(spint(A)) \subseteq f(A) \).
By hypothesis $f(\text{spint}(A))$ is an $\beta_\tau$ open set and $\text{spint}(f(A))$ is largest $\beta_\tau$ open set contained in $f(A)$ so $f(\text{spint}(A)) \subseteq \text{spint}(f(A))$.

Conversely, suppose $A$ is an tri open set in $X$. So $f(\text{spint}(A)) \subseteq \text{spint}(f(A))$.

Now since $A = \text{spint}(A)$ so $f(A) \subseteq \text{spint}(f(A))$. Therefore $f(A)$ is a $\beta_\tau$ open set in $Y$ and $f$ is $\beta_\tau$ open continuous function.

**Theorem 3.11:** A function $f:(X, P_1, P_2, P_3) \rightarrow (Y, W_1, W_2, W_3)$ is called $\beta_\tau$ closed continuous function if and only if $\text{spcl}(f(A)) \subseteq f(\text{spcl}(A))$, for every tri closed set $A$ in $X$.

**Proof:** Suppose that $f$ is a $\beta_\tau$ closed continuous function. since $A \subseteq \text{spcl}(A)$ so $f(A) \subseteq f(\text{spcl}(A))$. By hypothesis, $f(\text{spcl}(A))$ is a $\beta_\tau$ closed set and $\text{sccl}(f(A))$ is smallest $\beta_\tau$ closed set containing $f(A)$ so $\text{sccl}(f(A)) \subseteq f(\text{sccl}(A))$.

Conversely, suppose $A$ is an tri closed set in $X$. So $\text{sccl}(f(A)) \subseteq f(\text{spcl}(A))$.

Since $A = \text{spcl}(A)$ so $f(\text{spcl}(A)) \subseteq f(A)$. Therefore $f(A)$ is a $\beta_\tau$ closed set in $Y$ and $f$ is $\beta_\tau$ closed continuous function.

**Theorem 3.12:** Let $(X, T_1, T_2, T_3)$ and $(Y, W_1, W_2, W_3)$ be two tri topological spaces. Then, $f:X \rightarrow Y$ is $\beta_\tau$ continuous function if and only if $f^{-1}(V)$ is $\beta_\tau$ closed in $X$ whenever $V$ is tri closed in $Y$.

**Theorem 3.13:** Let $(X, T_1, T_2, T_3)$ and $(Y, W_1, W_2, W_3)$ be two tri topological spaces. Then, $f:X \rightarrow Y$ is $\beta_\tau$ continuous function if and only if $f(\text{spcl}(A)) \subseteq \text{spcl}(f(A)) \forall A \in X$.

**Proof:** Suppose $f:X \rightarrow Y$ is $\beta_\tau$ continuous function. Since $\text{spcl}(f(A))$ is tri closed in $Y$. Then by theorem (3.4) $f^{-1}[\text{spcl}(f(A))]$ is tri closed in $X$.

$$\text{spcl}(f^{-1}(\text{spcl}(A))) = f^{-1}(\text{spcl}(f(A)))$$

Now: $f(A) \subseteq \text{spcl}(f(A)), A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\text{spcl}(f(A)))$.

Then $\text{spcl}(f(A)) \subseteq \text{spcl}(f^{-1}(\text{spcl}(A))) = f^{-1}(\text{spcl}(A))$ by (1).

Then $f(\text{spcl}(A)) \subseteq \text{spcl}(f(A))$.

Conversely, let $f(\text{spcl}(A)) \subseteq \text{spcl}(f(A)) \forall A \in X$.

Let $F$ be $\beta_\tau$ closed set in $Y$, so that $\text{spcl}(F) = F$. Now $f^{-1}(F) \subseteq X$, by hypothesis,

$$f(\text{spcl}(f^{-1}(F))) \subseteq f(\text{spcl}(F)) \subseteq \text{spcl}(F) = F$$

Therefore $\text{spcl}(f^{-1}(F)) \subseteq f^{-1}(F)$. But $f^{-1}(F) \subseteq \text{spcl}(f^{-1}(F))$ always.

Hence $\text{spcl}(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is $\beta_\tau$ closed in $X$.

Hence by theorem (3.6) $f$ is $\beta_\tau$ continuous function.

**Theorem 3.14:** Let $(X, T_1, T_2, T_3)$ and $(Y, W_1, W_2, W_3)$ be two tri topological spaces. Then, $f:X \rightarrow Y$ is $\beta_\tau$ continuous function if and only if $\text{spcl}(f^{-1}(B)) \subseteq f^{-1}(\text{spcl}(B)) \forall B \subseteq Y$.

**Proof:** Suppose $f:X \rightarrow Y$ is $\beta_\tau$ continuous. Since $\text{spcl}(B)$ is $\beta_\tau$ closed in $Y$, then by theorem (3.6) $f^{-1}(\text{spcl}(B))$ is $\beta_\tau$ closed in $X$ and therefore, $\text{spcl}(f^{-1}(\text{spcl}(B))) = f^{-1}(\text{spcl}(B))$.

Now, $B \subseteq \text{spcl}(B)$, then $f^{-1}(B) \subseteq f^{-1}(\text{spcl}(B))$, then $\text{spcl}(f^{-1}(B)) \subseteq \text{spcl}(f^{-1}(\text{spcl}(B))) = f^{-1}(\text{spcl}(B))$ by (2).

Conversely: Let the condition hold and let $F$ be any tri closed set in $Y$ so that $\text{spcl}(F) = F$. By hypothesis, $\text{spcl}(f^{-1}(F)) \subseteq f^{-1}(\text{spcl}(F))$. But $f^{-1}(F) \subseteq \text{spcl}(f^{-1}(F))$ always. Hence $\text{spcl}(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is $\beta_\tau$ closed in $X$. It follows from theorem (3.6) that $f$ is $\beta_\tau$ continuous function.

**Theorem 3.15:** Let $(X, T_1, T_2, T_3)$ and $(Y, P_1, P_2, P_3)$ be two tri topological spaces. Then, $f:X \rightarrow Y$ is $\beta_\tau$ continuous open function if and only if $f^{-1}(\text{spint}(B)) \subseteq \text{spint}(f^{-1}(B)) \forall B \subseteq Y$.

**Proof:** Let $f:X \rightarrow Y$ be a $\beta_\tau$ continuous. Since $\text{spint}(B)$ is $\beta_\tau$ open in $Y$, then by theorem (3.3) $f^{-1}(\text{spint}(B))$ is $\beta_\tau$ open in $X$ and therefore, $\text{spint}(f^{-1}(\text{spint}(B))) = f^{-1}(\text{spint}(B))$.

Now, $\text{spint}(B) \subseteq B$, then $f^{-1}(\text{spint}(B)) \subseteq f^{-1}(B)$, then $\text{spint}(f^{-1}(\text{spint}(B))) \subseteq \text{spint}(f^{-1}(B))$ by (3).

Conversely: Let the condition hold and let $G$ be any $\beta_\tau$ open set in $Y$ so that $\text{spint}(G) = G$. By hypothesis, $f^{-1}(\text{spint}(G)) \subseteq \text{spint}(f^{-1}(G))$. Since $f^{-1}(\text{spint}(G)) = f^{-1}(G)$ then $f^{-1}(G) \subseteq \text{spint}(f^{-1}(G))$. But
REFERENCES


\[ \text{sp}(f^{-1}(G)) \subseteq f^{-1}(G) \text{ always and so sp}(f^{-1}(G)) = f^{-1}(G). \text{ Therefore } f^{-1}(G) \text{ is } \beta_T \text{ open in } X. \text{ Consequently by theorem (3.3) } f \text{ is } \beta_T \text{ continuous function.} \]

CONCLUSION: In this paper we studied new form of \( \beta_T \) open set in tri topological space. We also studied \( \beta_T \) continuous function in tri topological space. It is established that composition of any two \( \beta_T \) open sets is again a \( \beta_T \) open set in tri topological space. In \( \beta_T \) continuity, inverse image of every tri open is \( \beta_T \) open set.

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