ORIGINAL RESEARCH PAPER
Volume-6 | Issue-11 | November-2017 | ISSN No 2277 - 8179 | IF : 4.176 | IC Value : 78.46
INTERNATIONAL JOURNAL OF SCIENTIFIC RESEARCH

INTEGRAL FORMULAS INVOLVING PRODUCT OF SRIVASTAVA'S POLYNOMIAL AND GENERALIZED BESSEL MAITLAND FUNCTIONS

Mathematics

D.L. Suthar Department of Mathematics, Wollo University, Dessie, P.O. Box: 1145, South Wollo, Amhara Region, (Ethiopia).

G.V. Reddy Department of Mathematics, Wollo University, Dessie, P.O. Box: 1145, South Wollo, Amhara Region, (Ethiopia).

Nigussie Abeye Department of Mathematics, Wollo University, Dessie, P.O. Box: 1145, South Wollo, Amhara Region, (Ethiopia).

ABSTRACT

The aim of this paper is to establish two general finite integral formulas involving the product of generalized Bessel-Maitland functions $J_{v,q}^{\mu,\gamma}(z)$ and Srivastava polynomial $S_n^m(x)$ The result given in terms of generalized (Wright's) hypergeometric functions ${}_p\Psi_q$. These results are obtained with the help of finite integral due to Oberhettinger. Some interesting special cases of the main results are also considered. The results presented here are of general character and easily reducible to new and known integral formulae.

KEYWORDS

Oberhettinger integral formula, Srivastava polynomial, Gamma function, Bessel function, Generalized Bessel-Maitland function, Generalized Wright hypergeometric functions.

Mathematics Subject Classification: 26A33, 33B15, 33C10, 33C02.

1. Introductionand Preliminaries

In special function, one of the most important function (Bessel function) is widely used in physics and engineering; therefore, they are of interest to physicists and engineers as well as mathematician. In recent year, a remarkably large number of integral formulas involving a variety of special function have been developed by many authors as Agarwal et al. [2, 3], Brychkov[4], Choi et al. [5], Choi and Agarwal [6, 7], Manaria et al. [13, 14], Nisar et al. [16], Ramachandran et al. [20] and Suthar et al. [26]. We aim at presenting two generalized integral formulas involving the Bessel-Maitland function, which are expressed in term of the generalized (Wright's) hypergeometric.

For our purpose, we begin by recalling some known function and earlier works. The Bessel-Maitland function $J_v^\mu(z)$ defined by the following series representation by Merichev [12] as follow:

$$J_v^\mu(z)=\sum_{m=0}^{\infty}\frac{(-z)^m}{\Gamma(v+\mu m+1)\,m!}, \qquad \text{where } \mu>0;\ z\in\mathbb{C} \tag{1.1}$$

The generalized Bessel function of the form $J_{v,\sigma}^\mu(z)$ is defined by Jain and Agarwal [10] as follow:

$$J_{v,\sigma}^\mu(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2\sigma+2m}}{\Gamma(v+\sigma+\mu m+1)(\sigma+m+1)}, \tag{1.2}$$

where $z\in\mathbb{C}\setminus(-\infty,0];\mu>0, v, \sigma\in\mathbb{C}$.

Further, generalization of the generalized Bessel-Maitland function $J_{v,q}^{\mu,\gamma}(z)$ defined by Pathak [18] as follows:

$$J_{v,q}^{\mu,\gamma}(z)=\sum_{m=0}^{\infty}\frac{(\gamma)_{qm}(-z)^m}{\Gamma(v+\mu m+1)\,m!} \tag{1.3}$$

where $\mu,v,\gamma\in\mathbb{C}\ \Re(\mu)\geq 0,\ \Re(v)\geq -1,\ \Re(\gamma)\geq 0$ and $q\in(0,1)\cup\mathbb{N}$, and $(\gamma)_0 = 1,\ (\gamma)_{qm} = \frac{\Gamma(\gamma + qm)}{\Gamma(\gamma)}$ is known as generalized Pochhammer symbol defined by Mittag-Leffler [15].

From the generalization of the generalized Bessel-Maitland function (1.3), it is possible to find some special cases by giving particular values to the parameters μ, v, γ, q.

1) If $q=1, \gamma=1$ and v is replaced by $v+\sigma$ and z is replaced by $\frac{z^2}{4}$ in (1.3), then we obtain

$$J_{v+\sigma,\,1}^{\mu,1}\left(\frac{z^2}{4}\right) = \Gamma(\sigma+m+1)\left(\frac{z}{2}\right)^{-v-2\sigma} J_{v,\sigma}^{\mu}(z), \tag{1.4}$$

where $J_{v,\,\sigma}^{\mu}(z)$ denotes Bessel-Maitland function defined by Agarwal et al. [3].

2) If we replace μ by 1 and σ by $\frac{1}{2}$ in (1.4), we obtain

$$J_{v+\frac{1}{2},\,1}^{1,\,1}\left(\frac{z^2}{4}\right) = \Gamma\left(m+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{-v-1} H_v(z), \tag{1.5}$$

where $H_v(z)$ denotes Struve's function defined by Erdẻlyi et al. [8].

$$H_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(v+m+\frac{3}{2}\right)}. \tag{1.6}$$

3) If $q=0$, then (1.3) reduces to

$$J_{v,\,0}^{\mu,\,\gamma}(z) = J_v^{\mu}(z), \tag{1.7}$$

where $J_v^{\mu}(z)$ is generalized Bessel function defined by Agarwal [1].

4) If $q=0$, $\mu=1$ and z is replaced by $\frac{z^2}{4}$ then (1.3) reduces to

$$J_{v,\,0}^{1,\,\gamma}\left(\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^{-v} J_v(z), \tag{1.8}$$

where $J_v(z)$ is called Bessel's function of the first kind and of order v, where v is any non-negative constant.

5) If $q=0$ and v is replaced by $v-1$ and z is replaced by $-z$, then (1.3) reduces to

$$J_{v-1,\,0}^{\mu,\,\gamma}(-z) = \phi(\mu, v; z), \tag{1.9}$$

where $\phi(\mu, v; z)$ is known as Wright function, defined by Choi et al. [5].

6) If v is replaced by $v-1$ and z is replaced by $-z$, then (1.3) reduces to

$$J_{v-1,\,q}^{\mu,\,\gamma}(-z) = E_{\mu,\,v}^{\gamma,\,q}(z), \tag{1.10}$$

where $E_{\mu,\,v}^{\gamma,\,q}(z)$ is generalized Mittag -Leffler function , was given by Shukla and Prajapati [21].

7) If $q=1$, v is replaced by $v-1$ and z is replaced by $-z$, then (1.3) reduces to

$$J_{v-1,\,1}^{\mu,\,\gamma}(-z) = E_{\mu,\,v}^{\gamma}(z), \tag{1.11}$$

was introduced by Prabhakar [19].

8) If $q=1$, $\gamma=1$, v is replaced by $v-1$ and z is replaced by $-z$, (1.3) reduces to

$$J_{v-1,\,1}^{\mu,\,1}(-z)=E_{\mu,v}(z), \tag{1.12}$$

where $\mu\in\mathbb{C},\ \Re(\mu)>0,\ \Re(v)>0$, was studied by Wiman [27].

9) If $q=1$, $\gamma=1$, $v=0$ and z is replaced by $-z$, (1.3) reduces to

$$J_{0,\,1}^{\mu,\,1}(-z)=E_{\mu}(z). \tag{1.13}$$

where $\mu\in\mathbb{C},\Re(\mu)>0$, was introduced by Mittag-Leffler [15].

The generalized Wright hypergeometric function ${}_p\psi_q(z)$ (see, for detail, Srivastava and Karlsson [24]), for $z\in\mathbb{C}$ complex, $a_i,\ b_j\in\mathbb{C}$ and $\alpha_i,\ \beta_j\in\mathbb{R}$, where $(\alpha_i,\ \beta_j\neq 0;\ i=1,2,\ldots,p;\ j=1,2,\ldots,q)$, is defined as below:

$$ {}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,\ p}\\(b_j,\beta_j)_{1,\ q}\end{matrix}\Big|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)\,z^k}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)\ k!}, \tag{1.14}$$

Introduced by Wright [28], the generalized Wright function and proved several theorems on the asymptotic expansion of ${}_p\psi_q(z)$ for all values of the argument z, under the condition:

$$\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1. \tag{1.15}$$

It is noted that the generalized (Wright) hypergeometric function ${}_p\psi_q$ in (1.15) whose asymptotic expansion was investigated by Fox [9].

For this we recall following known functions. The Srivastava's polynomial defined by Srivastava [23]:

$$S_n^m[x]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,x^k \qquad (n=0,1,2,\ldots) \tag{1.16}$$

where m is an arbitrary positive integer and the coefficient $A_{n,k}\,(n,k\geq 0)$ are arbitrary constants, real or complex. The polynomial family $S_n^m[x]$ gives a number of known polynomials as its special cases on suitably specializing the coefficients $A_{n,k}$.

For our present investigation, we also need to recall the following Oberhettinger's integral formula defined by Oberhettinger [17]:

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx=2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}. \tag{1.17}$$

provided $0<\Re(\mu)<\Re(\lambda)$.

2. Main Results

In this section, we established two generalized integral formulae involving product of generalized Bessel-Maitland function (1.3) and Srivastava polynomials (1.16) are established, which are expressed in term of generalized (Wright) hypergeometric functions.

Theorem 1: *The following integral formula holds true for* $\delta,\lambda,\mu,v,\gamma\in\mathbb{C}$ *with* $\mathrm{Re}(v)\geq -1,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,\ 0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $x>0,\ n,k\geq 0$, *we have*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) J_{v,q}^{\mu,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\,\frac{\Gamma(2\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{matrix}(\gamma,q),\,(\lambda+k+1,1),\,(\lambda+k-\delta,1);\\ \\ (v+1,\mu),\,(\lambda+k,1),\,(1+\lambda+k+\delta,1);\end{matrix}\;-\frac{y}{a}\right] \quad (2.1)$$

Theorem 2: *The following integral formula holds truefor* $\delta,\lambda,\mu,v,\gamma\in\mathbb{C}$ *with* $\mathrm{Re}(v)\geq -1,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,\ 0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $x>0,\ n,k\geq 0$ *, we have*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) J_{v,q}^{\mu,\gamma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\,\frac{\Gamma(\lambda-\delta)}{\Gamma(\gamma)}\;{}_3\Psi_3\left[\begin{matrix}(\gamma,q),\,(\lambda+k+1,1),\,(2\delta+2k,2);\\ \\ (v+1,\mu),\,(\lambda+k,1),\,(1+\lambda+\delta+2k,2);\end{matrix}\;-\frac{y}{2}\right] \quad (2.2)$$

Proof: By making use of product of (1.3) and (1.16) in the integrand of (2.1) and then interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions in Theorem 1, we get

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) J_{v,q}^{\mu,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\sum_{l=0}^{\infty}\frac{(\gamma)_{ql}(-y)^l}{\Gamma(v+\mu l+1)\,l!}\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-k-l}dx,$$

By considering the condition given in Theorem 1, since $x>0,\ n,k\geq 0,\ \mathrm{Re}(v)\geq -1,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,\ 0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $q\in(0,1)\cup\mathbb{N}$ and applying (1.17),

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k-l}(-y)^l\,\frac{\Gamma(2\delta)}{\Gamma(\gamma)}\sum_{l=0}^{\infty}\frac{\Gamma(\gamma+ql)}{\Gamma(v+\mu l+1)}\frac{\Gamma(\lambda+k+l+1)\Gamma(\lambda+k+l-\delta)}{\Gamma(\lambda+k+l)\Gamma(1+\lambda+k+l+\delta)\,l!},$$

which upon using the definition (1.14), we get the desired result (2.1).

By similar manner as in proof of Theorem 1, we can also prove the integral formula (2.2).

3. Special Cases

In this section, we represent certain cases of generalized form of Bessel-Maitland function (1.3).

On setting $\gamma=1,\ v=v+\sigma$, $q=1$ and z is replaced by $(z^2/4)$, in theorem 1 and theorem 2 and making use of the relation (1.4), then the generalized Bessel-Maitland function will have following relation with Bessel-Maitland function as follows:

Corollary 1. *Let the condition of* $\delta,\lambda,\mu,v\in\mathbb{C},\ x>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(v)\geq -1,\ \mathrm{Re}(\mu)>0,\ 0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $n,k\geq 0$, *be satisfied, then the following integral formula holds true*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) J_{v,\sigma}^{\mu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\, \Gamma(2\delta)\; {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 2),\ (\lambda+k-\delta,2),(1,1);\\ (v+\sigma+1,\mu),\ (\lambda+k,2),\ (1+\lambda+k+\delta,2);\end{matrix}\ -\frac{y^2}{4a^2}\right] \qquad (3.1)$$

Corollary 2 *Let the condition of* $\delta,\lambda,\mu, v\in \mathbb{C},\ x>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(v)\geq -1,\ \mathrm{Re}(\mu)>0,$ $0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) J_{v,\sigma}^{\mu}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\, \Gamma(\lambda-\delta)\; {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 2),\ (2\delta+2k,4),\ (1,\ 1);\\ (v+\sigma+1,\mu),\ (\lambda+k,\ 2),\ (1+\lambda+\delta+2k,\ 4);\end{matrix}\ -\frac{y^2}{16}\right] \qquad (3.2)$$

On setting $\mu=1, \sigma=1/2$ in (3.1) and (3.2) and using the relation (1.5), then we get the integral formulas involving the Struve's function $H_v(z)$ as follows:

Corollary 3 *Let the condition of* $\delta,\lambda, v\in \mathbb{C},\ x>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,$ $0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) H_v\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\, \Gamma(2\delta)\; {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 2),\ (\lambda+k-\delta,2),(1,1);\\ \left(v+\frac{3}{2},1\right),\ (\lambda+k,\ 2),\ (1+\lambda+k+\delta,2);\end{matrix}\ -\frac{y^2}{4a^2}\right] \qquad (3.3)$$

Corollary 4 *Let the condition of* $\delta,\lambda, v\in \mathbb{C},\ x>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\mu)>0,$ $0<\mathrm{Re}(\delta)<\mathrm{Re}(\lambda)$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) H_v\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\, \Gamma(\lambda-\delta)\; {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 2),\ (2\delta+2k,4),\ (1,\ 1);\\ \left(v+\frac{3}{2},\ 1\right),\ (\lambda+k,\ 2),\ (1+\lambda+\delta+2k,\ 4);\end{matrix}\ -\frac{y^2}{16}\right] \qquad (3.4)$$

On setting $q=0$ in theorem 1 and theorem 2 and making use of the relation (1.7), then the generalized Bessel–Maitland function $J_{v,q}^{\mu,\gamma}(z)$ will have the following relation with Bessel-Maitland function $J_v^{\mu}(z)$ as follows:

Corollary 5. *Let the condition of* $\delta,\lambda,\mu,v\in\mathbb{C},\ x>0,\quad \mathrm{Re}(\delta)>0,\quad \mathrm{Re}(\mu)>0,$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) J_v^{\mu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,y^k\,2^{1-\delta}\,a^{\delta-\lambda-k}\,\Gamma(2\delta)\;{}_2\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 1),\ (\lambda+k-\delta,1);\\ \\ (v+1,\mu),\ (\lambda+k,1),\ (1+\lambda+k+\delta,1);\end{matrix}\ \middle|\ -\frac{y}{a}\right] \tag{3.5}$$

Corollary 6. *Let the condition of* $\delta,\lambda,\mu,v\in\mathbb{C},\ x>0,\quad \mathrm{Re}(\delta)>0,\quad \mathrm{Re}(\mu)>0,$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) J_v^{\mu}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,y^k\,2^{1-\delta-k}\,a^{\delta-\lambda}\,\Gamma(\lambda-\delta)\;{}_2\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 1),\ (2\delta+2k,2);\\ \\ (v+1,\mu),\ (\lambda+k,\ 1),\ (1+\lambda+\delta+2k,\ 2);\end{matrix}\ \middle|\ -\frac{y}{2}\right] \tag{3.6}$$

On setting $q=0,\ \mu=1$ and z is replaced by $(z^2/4)$, in theorem 1 and theorem 2 and making use of the relation (1.8), we obtain the following integral formulas involving the ordinary Bessel function as follows:

Corollary 7. *Let the condition* $\delta,\lambda\in\mathbb{C},\ x>0,\quad \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\lambda)>0,$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-2v} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) J_v\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,y^{v+k}\,2^{1-\delta-v}\,a^{\delta-\lambda-v-k}\,\Gamma(2\delta)$$

$$\times\,{}_2\Psi_3\left[\begin{matrix}(\lambda+2v+k+1,2),\ (\lambda+2v+k-\delta,2);\\ \\ (v+1,1),\ (\lambda+2v+k,2),\ (1+\lambda+2v+k+\delta,2);\end{matrix}\ \middle|\ -\frac{y^2}{4a^2}\right] \tag{3.7}$$

Corollary 8. *Let the condition* $\delta,\lambda\in\mathbb{C},\ x>0,\quad \mathrm{Re}(\delta)>0,\quad \mathrm{Re}(\lambda)>0,$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-2\nu} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) J_\nu\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda-2\nu}\,\Gamma(\lambda+2\nu-\delta)$$

$$\times\, {}_2\Psi_3\left[\begin{matrix}(\lambda+2\nu+k+1,\ 2),\ (2\delta+2k,4);\\ (\nu+1,1),\ (\lambda+2\nu+k,\ 2),\ (1+\lambda+2\nu+\delta+2k,\ 4);\end{matrix}\ -\frac{y^2}{16}\right] \tag{3.8}$$

On setting $q=0$ and ν is replaced by $\nu-1$ and z is replaced by $-z$, in theorem 1 and theorem 2 and making use of the relation (1.9), we obtain the following integral formulas involving the Weight function as follows:

Corollary 9 *Let the condition* $\delta,\lambda\in\mathbb{C},\ x>0,\quad \mathrm{Re}(\delta)>0,\quad \mathrm{Re}(\lambda)>0,$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)\phi\left(\mu,\nu;\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx`$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\,\Gamma(2\delta)\ {}_2\Psi_3\left[\begin{matrix}(\lambda+k+1,1),\ (\lambda+k-\delta,1);\\ (\nu,\ \mu),\ (\lambda+k,1),\ (1+\lambda+k+\delta,1);\end{matrix}\ -\frac{y}{a}\right] \tag{3.9}$$

Corollary 10. *Let the condition* $\delta,\lambda\in\mathbb{C},\ x>0,\quad \mathrm{Re}(\delta)>0,\quad \mathrm{Re}(\lambda)>0,$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)\phi\left(\mu,\nu;\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\,\Gamma(\lambda-\delta)\ {}_2\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 1),\ (2\delta+2k,2);\\ (\nu,1),\ (\lambda+k,\ 1),\ (1+\lambda+\delta+2k,\ 2);\end{matrix}\ -\frac{y}{2}\right] \tag{3.10}$$

On setting ν by $\nu-1$ and z is replaced by $-z$, in theorem 1 and theorem 2 and making use of the relation (1.10), we obtain the following integral formulas involving the generalized Mittag- Leffler function as follows:

Corollary 11. *Let the condition of* $\delta,\ \lambda,\mu,\ \nu,\ \gamma\in\mathbb{C},\ \ \Re(\delta)>0, \Re(\lambda)>0\ \ \Re(\mu)>0, \Re(\gamma)>0,$ $\Re(\nu)\geq -1,\ \ q\in(0,1)\cup\mathbb{N}$ *and* $n,k\geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)E_{\mu,\nu}^{\gamma,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\,\Gamma(2\delta)\ {}_3\Psi_3\left[\begin{matrix}(\gamma,q),\ (\lambda+k+1,1),\ (\lambda+k-\delta,1);\\ (\nu,\mu),\ (\lambda+k,1),\ (1+\lambda+k+\delta,1);\end{matrix}\ -\frac{y}{a}\right] \tag{3.11}$$

Corollary 12. *Let the condition of* $\delta, \lambda, \mu, v, \gamma \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0, \Re(\gamma) > 0$, $\Re(v) \geq -1$, $q \in (0,1) \cup \mathbb{N}$ *and* $n, k \geq 0$, *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu,v}^{\gamma,q}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\, \Gamma(\lambda-\delta)\; {}_3\Psi_3\left[\begin{matrix}(\gamma, q), (\lambda+k+1, 1), (2\delta+2k, 2); \\ (v, \mu), (\lambda+k, 1), (1+\lambda+\delta+2k, 2);\end{matrix} \middle| \frac{y}{2}\right] \tag{3.12}$$

On setting $q=1, v$ by $v-1$ and z is replaced by $-z$, in theorem 1 and theorem 2 and making use of the relation (1.11), we obtain the following integral formulas involving the generalized Mittag- Leffler function as follows:

Corollary 13. *Let the condition of* $\delta, \lambda, \mu, v, \gamma \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0, \Re(\gamma) > 0$, $\Re(v) \geq -1$, *and* $n, k \geq 0$, *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu,v}^{\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\, \Gamma(2\delta)\; {}_3\Psi_3\left[\begin{matrix}(\gamma,1), (\lambda+k+1, 1), (\lambda+k-\delta,1); \\ (v,\mu), (\lambda+k,1), (1+\lambda+k+\delta,1);\end{matrix} \middle| \frac{y}{a}\right] \tag{3.13}$$

Corollary 14. *Let the condition of* $\delta, \lambda, \mu, v, \gamma \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0, \Re(\gamma) > 0$, $\Re(v) \geq -1$, *and* $n, k \geq 0$, *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu,v}^{\gamma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\, \Gamma(\lambda-\delta)\; {}_3\Psi_3\left[\begin{matrix}(\gamma, 1), (\lambda+k+1, 1), (2\delta+2k, 2); \\ (v, \mu), (\lambda+k, 1), (1+\lambda+\delta+2k, 2);\end{matrix} \middle| \frac{y}{2}\right] \tag{3.14}$$

On setting $\gamma=1, v$ by $v-1$ and z is replaced by $-z$, in theorem 1 and theorem 2 and making use of the relation (1.12), we obtain the following integral formulas involving the generalized Mittag-Leffler function as follows:

Corollary 15. *Let the condition of* $\delta, \lambda, \mu, v \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0$, *and* $n, k \geq 0$, *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu,v}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\, \Gamma(2\delta)\; {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1, 1), (\lambda+k-\delta,1), (1,1); \\ (v,\mu), (\lambda+k,1), (1+\lambda+k+\delta,1);\end{matrix} \middle| \frac{y}{a}\right] \tag{3.15}$$

Corollary 16. *Let the condition of* $\delta, \lambda, \mu, v \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0,$ *and* $n, k \geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu,v}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\, \Gamma(\lambda-\delta)\, {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 1),\ (2\delta+2k,2),\ (1,\ 1); \\ (v,\ \mu),\ (\lambda+k,\ 1),\ (1+\lambda+\delta+2k,\ 2);\end{matrix}\ \middle|\ \frac{y}{2}\right] \quad (3.16)$$

On setting $\gamma=1,\ v=0,\ q=1$ and z is replaced by $-z$, in theorem 1 and theorem 2 and making use of the relation (1.13), we obtain the following integral formulas involving the Mittag-Leffler function as follows:

Corollary 17. *Let the condition of* $\delta, \lambda, \mu \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0,$ *and* $n, k \geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta}\, a^{\delta-\lambda-k}\, \Gamma(2\delta)\, {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,1),\ (\lambda+k-\delta,1),\ (1,1); \\ (1,\mu),\ (\lambda+k,1),\ (1+\lambda+k+\delta,1);\end{matrix}\ \middle|\ \frac{y}{a}\right] \quad (3.17)$$

Corollary 18 *Let the condition of* $\delta, \lambda, \mu \in \mathbb{C}$, $\Re(\delta) > 0, \Re(\lambda) > 0$ $\Re(\mu) > 0,$ *and* $n, k \geq 0,$ *be satisfied, then the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} S_n^m\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) E_{\mu}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, y^k\, 2^{1-\delta-k}\, a^{\delta-\lambda}\, \Gamma(\lambda-\delta)\, {}_3\Psi_3\left[\begin{matrix}(\lambda+k+1,\ 1),\ (2\delta+2k,2),\ (1,\ 1); \\ (1,\ \mu),\ (\lambda+k,\ 1),\ (1+\lambda+\delta+2k,\ 2);\end{matrix}\ \middle|\ \frac{y}{2}\right] \quad (3.18)$$

Remark: If we set $n=0$, then we observe that the general class of polynomials $S_n^m[x]$ reduces to unity, i.e. $S_0^m[x] \to 1$, and we get the following known results due to Khan and Kashmin [11].

4. Concluding remarks

In the present paper, we investigate new integrals involving the generalized Bessel-Maitland function $J_{v,q}^{\mu,\gamma}(z)$ in terms of generalized (Wright) hypergeometric functions. Certain special cases of integrals involving generalized Bessel Maitland function have been investigated in the literature by a number of authors ([1-7, 11, 13, 14, 25]) with different arguments. Therefore the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements. In this sequel, one can obtain integral representation of more generalized special function, which has much application in physics and engineering Science.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

REFERENCES

[1] Agarwal P. (2015). Pathway fractional integral formulas involving Bessel function of the first kind, Advanced Studies in Contemporary Mathematics, 25(1), 221-231.
[2] Agarwal P., Jain S., Agarwal S., & Nagpal M. (2014). On a new class of integrals involving Bessel functions of the first kind, Communication in Numerical Analysis, 1-7.
[3] Agarwal P., Jain S., Chand M., Dwivedi S.K. & Kumar S. (2014). Bessel functions associated with Saigo-Maeda fractional derivatives operators, Journal of Fractional Calculus and Applications, 5(2), 102-112.
[4] Brychkov Y.A. (2008). Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas, CRC Press, Taylor and Francis Group, Boca Raton, London, and New York.
[5] Choi J., Agarwal P., Mathur S. & Purohit S.D. (2014). Certain new integral formulas involving the generalized Bessel functions, Bulletin of the Korean Mathematical Society, 51(4), 995-1003.
[6] Choi J. & Agarwal P. (2013). Certain unified integrals involving a product of Bessel functions of first kind, Honam Mathematical Journal, 35 (4), 667-677.
[7] Choi J. & Agarwal P. (2013). Certain unified integrals associated with Bessel functions, Boundary Value Problems, doi:10.1186/1687-2770-2013-95.
[8] Erdélyi A., Magnus W., Oberhettinger F. & Tricomi F. (1954). Table of integral transforms, Vol. II, McGraw-Hill, New York.
[9] Fox C. (1928). The asymptotic expansion of generalized hypergeometric functions, Proceedings of the London Mathematical Society, 27(2), 389-400.
[10] Jain S. & Agarwal P. (2015). A new class of integral relation involving general class of Polynomials and I- function, Walailak Journal of Science and Technology, 12(11), 1009-1018.
[11] Khan N.U. & Kashmin T. (2016). Some integrals for the generalized Bessel Maitland functions, Electronic Journal of Mathematical Analysis and Applications, 4(2), 139-149.
[12] Marichev O. I. (1983). Handbook of integral transform and Higher transcendental functions, Ellis, Harwood, chichester (John Wiley and Sons); New York.
[13] Manaria N., Nisar K.S. & Purohit S.D. (2016). On a new class of integrals involving product of generalized Bessel function of the first kind and general class of polynomials, Acta Universitatis Apulensis , 46, 97-105.
[14] Manaria N., Purohit S.D. & Parmar R.K. (2016). On a new class of integrals involving generalized Mittag-Leffler function, Survey of mathematics and its applications, 11, 1-9.
[15] Mitag-Leffler G.M. (1903). Sur la nouvelle function E_(x), Comptes rendus de l'Académie des Sciences Paris, 137, 554-558.
[16] Nisar K.S., Parmar R.K. & Abusufian A.H. (2016). Certain new unified integrals with the generalized k-Bessel function, Far East Journal of Mathematical Sciences, 100, 1533-1544.
[17] Oberhettinger F. (1974). Tables of Mellin Transform, Springer- Verlag, New York.
[18] Pathak R.S. (1996), Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations, The Proceedings of the National Academy of Sciences, India, Section A, 36(1), 81-86.
[19] Prabhaker T.R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Mathematical Journal , 19, 7-15.
[20] Ramachandran U.K., Rakha M.A. & Rathie A.K. (2015). Certain new unified integrals associated with the product of generalized Bessel functions, Communication in Numerical Analysis, 2, 104-114.
[21] Shukla A.K. & Prajapati J.C. (2007). On a generalization of Mittag-Leffler function and its properties, Journal of Mathematical Analysis and Applications, 336 (2), 797-811.
[22] Singh M., Khan M.A. & Khan A.H. (2014). On some properties of a generalization of Bessel-Maitland function, Journal of Mathematics trends and Technology, 14(1), 46-54.
[23] Srivastava H.M. (1972). A contour integral involving Fox's H-function, Indian Journal of Mathematics, 14, 1-6.
[24] Srivastava H.M. & Karlsson P.W. (1985). Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto.
[25] Suthar D.L., Haile Habenom Anteneh (2016). Integrals Involving Generalized Bessel-Maitland Function, Journal of Science and Arts, 37(4), 357-362.
[26] Suthar D.L., Reddy G.V. & Temesgen Tsegaye (2017). Unified Integrals Formulas Involving Product of Srivastava's Polynomials and Generalized Bessel-Maitland Function, International Journal of Scientific research, 6(2), 708-710.
[27] Wiman A. (1995). Uber de fundamental satz in der theorie der funktionen, Acta Mathematica, 29, 191-201.
[28] Wright E.M. (1935). The asymptotic expansion of the generalized hypergeometric functions, Journal of the London Mathematical Society, 10, 286-293.