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TOTAL EDGE IRREGULARITY STRENGTH OF GENERALIZED UNIFORM THETA GRAPH

Mathematics

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ABSTRACT

Given a graph *G(V,E)* a labeling $\partial: V \cup E \to \{1,2,...,k\}$ is called an edge irregular total *k*- labeling if for every pair of distinct edges *uv* and *xy*, $\partial(u)+\partial(uv)+\partial(v) \neq \partial(x)+\partial(xy)+\partial(y)$. The minimum *k* for which *G* has an edge irregular total *k*-labeling is called the *total edge irregularity strength* of *G*. The total edge irregular strength of *G* is denoted by tes(*G*). In this paper we consider uniform theta graph and prove that it is total edge irregular. We also discuss its bound.

KEYWORDS

Edge Irregular Labeling, Irregular Total Labeling, Total Edge Irregularity Strength, Series-parallel Graph, Theta Graph, Uniform Theta Graph.

1. Introduction

In the mathematical discipline of graph theory, a graph labeling is the assignment of labels, traditionally represented by integers, to the edges or vertices, or both, of a graph. Formally, given a graph *G*, a vertex labeling is a function mapping vertices of *G* to a set of "labels". A graph with such a function defined is called a *vertex-labeled* graph. Likewise, an edge labeling is a function mapping edges of *G* to a set of "labels". In this case, *G* is called an *edge-labeled* graph. When the edge labels are members of an ordered set (e.g., the real numbers), it may be called a *weighted graph*.

For a graph *G*(*V,E*), *E* not empty, it has been proved in [1] that $\lceil(|E|+2)/3\rceil \le tes(G) \le |E|$; $tes(G) \ge \lceil(\Delta(G)+1)/2\rceil$ and $tes(G) \le |E|-\Delta(G)$, if $\Delta(G) \le (|E|-1)/2$. They also proved that $tes(Pn) = tes(Cn) = \lceil(n+2)/3\rceil$; $tes(Wn) = \lceil(2n+2)/3\rceil$ and $tes(C_3^n)$ (friendship graph) $= \lceil(3n+2)/3\rceil$. Jendrol, Miškuf and Soták [5] (see also [6]) proved: $tes(K_5) = 5$; for $n \ge 6$, $tes(Kn) = \lceil(n^2-n+4)/6\rceil$; and that $tes(Km,n) = \lceil(mn+2)/3\rceil$.

Jendrol et al. [5] conjectured that for any graph G other than K_5, $tes(G) = \max\{\lceil(\Delta(G)+1)/2\rceil, \lceil(|E|+2)/3\rceil\}$. Ivančo and Jendrol [3] proved that this conjecture is true for all trees. Jendrol, Miškuf and Soták [5] proved the conjecture for complete graphs and complete bipartite graphs.

Theorem [1] Let *G* be a graph with *m* edges. Then $tes(G) \ge \lceil(m+2)/3\rceil$.

The above theorem motivates us to study the problem of proving the bound on *tes* is sharp for the *Generalized Uniform Theta Graph*.

1.1. Generalized Uniform Theta Graph

In graph theory, series-parallel graphs or *sp* graphs are graphs with two distinguished vertices called terminals, formed recursively by two simple composition operations. This classical definition justifies another name of these graphs, 2-terminal sp graphs, since we assume that every such graph has two nodes distinguished as poles denoted by S (for South) and N (for North). They can be used to model series and parallel electric circuits.

There are several ways to define series-parallel graphs. The following definition basically follows the one used by David Eppstein [2].

1.2 Definition

A *sp* graph *G* with poles *S* and N is defined as either

(i) an edge (*S, N*)

or can be constructed as in (ii) or (iii)

(ii) *G* is a parallel composition of atleast two sp graphs $G_1, G_2, ..., G_l$ ($l \ge 2$), denoted by *G* =

$\|G_1\| \| G_2 \| ... \|Gl\|$. This operation identifies the South poles S_i of the component graphs

into the South pole *S* of *G*, and similarly the North pole N_i become *N* of *G*.

(iii) *G* is a series composition of atleast two sp graphs $G_1, G_2, ..., Gk$ ($k \ge 2$), denoted by *G* =

$G_1{}^\circ G_2{}^\circ ...{}^\circ G_k$.

This operation identifies *Ni* and S_i+1 for *i* =1,..., *k*-1 and assigns S_1 to *S* and N_k to *N*.

1.3 Definition

A generalized theta graph *Θ(n, m)* or simply a theta graph with *n* vertices has two vertices N and *S* of degree *m* such that every other vertex is of degree 2 and lies in one of the *m* paths joining the vertices *N* and *S*. The two vertices *N* and S are called North Pole and South Pole respectively. A path between the South Pole and North Pole is called a longitude. A longitude is denoted by *L*. In the literature *Θ(n,3)* is called a theta graph.

A theta graph *Θ(n,l)* is said to be uniform if $|L_1|=|L_2|=...=|Ll|$, where *Li* is a longitude of *Θ*(*n*,*l*).

In [5] we had considered series-parallel graph of uniform theta graphs and proved that they are total edge irregular and its *tes* value is sharp for $l \ge 2$.

We had also proved the following results:

Lemma 1 $tes(sp(3,1,2)) = 5$.

Theorem 1 *Let* $sp(3,r,2)$ be a series parallel graph. Then $tes(sp(3,r,2)) = \lceil(6(r+1)+2)/3\rceil, r \ge 1$.

Lemma 2 $tes(sp(3,1,3)) = 7$.

Theorem 2 Let $sp(3,r,3)$ be a series parallel graph. Then $tes(sp(3,r,3)) = \lceil(9(r+1)+2)/3\rceil, r \ge 1$.

Theorem 3 Let $sp(m,r,l)$, $l \ge 2$ be a series parallel graph. Then $tes(sp(m,r,l)) = \lceil(lm(r+1)+2)/3\rceil, r \ge 1$.

In this paper we study the special case when $l = 1$ and obtain the sharp bound for sp(3, *r*, 1) and increase the bound by 1 for sp(*m, r, 1*), m > 3. When l =1, sp(*m, r, l*) is reduced as uniform theta graph, where uniform theta graph is denoted as follows.

1.4 Notation

In literature, for $m \ge 3, r \ge 1$, uniform theta graph with *m* longitudes and each longitude having *r* vertices excluding North Pole and South Pole is denoted by $\Theta(rm+2, m, r)$.

$\Theta(rm+2, m, r)$ has $m(r+1)$ edges and $mr+2$ vertices. As our study is on series composition of uniform theta graphs we shall use the following notation sp(*m, r, 1*) to denote a uniform theta graph as said above. We now prove the following lemma before stating the theorem for sp(*m, r,1*) for *m* = 3.

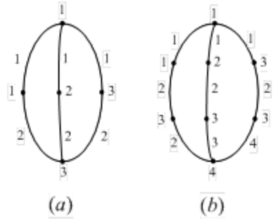

Fig 1. (*a*) $tes(sp(3,1,1)) = 3$ (*b*) $tes(sp(3,2,1)) = 4$

1. Main Results

1.1 Lemma $tes(sp(3,1,1)) = 3$.

Proof : Let $sp(3,1,1)$ be labeled as in Figure 1(*a*). It is easy to check that $tes(sp(3,1,1)) = 3$.

The following algorithm yields the total edge irregularity strength of $sp(3, r, 1)$.

Procedure $tes(sp(3,r,1))$

Input : Uniform theta graph, $sp(3,r,1)$, $r \geq 2$.

Algorithm:

Let s® $= tes(sp(3,r,1))$.

The vertices at level r in $sp(3,r,1)$ is the vertex at level r in $sp(3,r-1,1)$.

(1) Label the vertices and edges of $sp(3,1,1)$ as in Lemma 2.1.

(2) Having labeled $sp(3,1,1)$, label $sp(3,r,1)$, $r \geq 2$ as follows:

(i) Label vertices and edges at level r as in $sp(3,1,1)$.

(ii) Label the south pole as the *tes* value of $sp(3,r,1)$.

(iii) Having labeled the south pole, the edges connecting south pole to vertices at level r are

labeled from left to right as, $2s(r-1) - s(r)$, $2s(r-1) - s(r) + 1$ and $2s(r-1) - s(r) + 2$.

End Procedure $tes(sp(3,r,1))$.

Output: $tes(sp(3,r,1)) = \lceil (3(r+1)+2)/$

Proof of Correctness:

We prove the result by induction on r. When $r = 2$, the result is true. Assume the result for r -1. Consider $sp(3,r,1)$. Since the labeling of $sp(3,r-1,1)$ is an edge irregular k-labeling, it is clear that the labeling of vertices and edges of $sp(3,r,1)$ by using step 2 of above algorithm is also an edge irregular k-labeling. We know by actual verification that the edge sum labels obtained in Lemma 2.1 are consecutive and distinct. Hence the edge sum labels in $sp(3,r,1)$

are also distinct. Labeling of $sp(3,2,1)$ is shown in Figure 1(*b*).

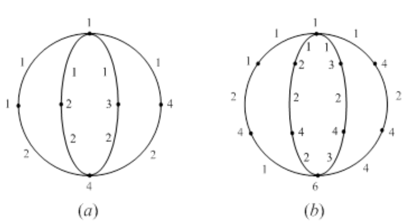

Fig 2. (*a*) $tes(sp(4,1,1)) = 4$ (*b*) $tes(sp(4,2,1)) = 6$

2.2 Theorem

Let $sp(3,r,1)$, for $r \geq 1$, denote the uniform theta graph with 3 longitudes and each longitude having r vertices. Then $tes(sp(3,r,1)) = \lceil (3r+5)/3 \rceil$.

We now discuss for the case m = 4 and hence conclude finally for $sp(m,r,1)$ which is stated as the main result of this section.

2.3 Lemma $tes(sp(4,1,1)) = 4$.

Proof : Let $sp(4,1,1)$ be labeled as in Figure 2(*a*). It is easy to check that $tes(sp(4,1,1)) = 4$.

The following was observed as we labeled $sp(4,1,1)$, which we state as a remark.

Remark: We noted that if $tes(sp(4,1,1)) = 5$, the *tes* value of $sp(4,2,1)$, would require 2 more than the minimum *tes* value expected for $sp(m,r,1)$ which will not be an interesting result. Hence we have considered $tes(sp(4,1,1)) = 4$ which is the *tes* value expected and have proceeded further with the result to prove that $tes(sp(m,r,1)) \leq \lceil (|E|+2)/3 \rceil + 1$, for $m \geq 4$ and $r \geq 2$, with base case $sp(m,2,1)$.

2.4 Lemma $tes(sp(4,2,1)) = 6$.

Proof: Let $sp(4,2,1)$ be labeled as in Figure 2(*b*). It is easy to check that $tes(sp(4,2,1)) = 6$.

We now proceed to prove the result for $sp(4,r,1)$, $r \geq 3$.

Procedure $tes(sp(4,r,1))$

Input :

Uniform theta graph, $sp(4,r,1)$, $r \geq 3$.

Algorithm:

Let k® $= tes(sp(4,r,1))$.

(1) Label the vertices and edges of $sp(4,2,1)$ as in Lemma 2.4.

(2) Having labeled $sp(4,2,1)$, label $sp(4,r,1)$, $r \geq 3$ as follows:

(i) Label vertices and edges at level r of $sp(4,r,1)$ as in $sp(4,2,1)$.

(ii) Label the south pole at level $r+1$ as the *tes* value of $sp(4,r,1)$.

(iii) Having labeled the south pole, the edges connecting south pole to the vertices at level r

are labeled from left to right as, $2k(r-1) - k(r) -1$, $2k(r-1) - k(r)$, $2k(r-1) - k(r)+1$ and $2k(r-1)-k(r)+2$.

End Procedure $tes(sp(4,r,1))$.

Output: $tes(sp(4,r,1)) \leq \lceil (4(r+1)+2)/3 \rceil + 1$.

Proof of Correctness:

We prove the result by induction on r. When $r = 2$, the result is true. Assume the result for $r+1$. Consider $sp(4,r,1)$. Since the labeling of $sp(4,r,1)$ is an edge irregular k-labeling, it is clear that the labeling of $sp(4,r+1,1)$ by step 2(iii) is also an edge irregular k-labeling. Since the labels are consecutive, the edge sum labels of $sp(4,r+1,1)$ are also consecutive integers which are clearly distinct. Labeling of $sp(4,3,1)$ and $sp(4,4,1)$ are shown in Figure 3(*a*) and (*b*).

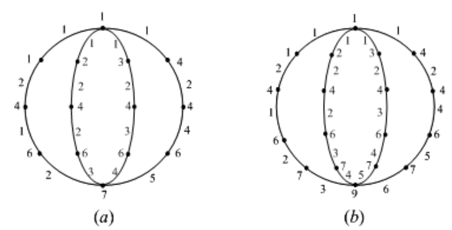

Fig 3. (*a*) $tes(sp(4,3,1)) = 7$ (*b*) $tes(sp(4,4,1)) = 9$

2.5 Theorem

Let $sp(4,r,1)$, be a uniform theta graph. Then

$$\lceil (4(r+1)+2)/3 \rceil \leq tes(sp(4,r,1)) \leq \lceil (4(r+1)+2)/3 \rceil + 1, r \geq 2.$$

The above result can be extended to any finite value of m.

2.6 Theorem

Let $sp(m,r,1)$, be a uniform theta graph. Then

$$\lceil (m(r+1)+2)/3 \rceil \leq tes(sp(m,r,1)) \leq \lceil (m(r+1)+2)/3 \rceil + 1, r \geq 2, m \geq 4.$$

Conclusion

In this paper we have considered the special case $l = 1$ of series parallel graph $sp(m,r,l)$ and proved that the lower bound for $sp(m,r,l)$ differs from the exact value by 1 for $r \geq 2$ and $m \geq 4$. Further our study is extended to total H- irregularity strength of graphs.

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