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SOME PROPERTIES OF GENERALIZED HYPERGEOMETRIC FUNCTION WITH LAGUERRE POLYNOMIAL

Mathematics

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ABSTRACT

In present paper, we obtain functions $R_t(1,\lambda+\nu,-n)$ and $R_t(1,\nu-\lambda,-n)$ by using generalized hypergeometric function. A recurrence relation, integral representation of the Generalized Hypergeometric Function ${}_2R_1(a,b;c;\tau;z)$ and some special cases have also been discussed.
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KEYWORDS

Generalized hypergeometric function, Integral representation, fractional integral and differential operators Laguerre Polynomial.

1. INTRODUCTION & PRILIMINARIES:

In many areas of applied mathematics and in particular, in applied analysis and fractional differential equations various types of special functions become essential tools for scientists and engineers. Further studying of the special functions is prospective and very useful for different branches of science. The diversity of the problems generating special functions has led to a quick in-crease in a number of special functions used in applications, The special functions play very important role , particularly the hypergeometric function in solving various numerical problems of mathematical physics, engineering and applied mathematics, is well known([4], [6], [9], [15]). This fact has inspired many mathematicians for investigations of several generalization of hypergeometric function ([5], [11], [16], [17], [18], [19], [20]), S. B. Rao, Amit D. Patel, Jyotindra C. Prajapati and Ajay K. Shukla [22] defines "Some Properties of Generalized Hypergeometric Functions" are carried out the as a special case in this paper as a particular case.

The Gauss Hypergeometric function is defined [12] as,

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!} \qquad (1.1)$$

where ($|z|<1, c\neq 0,-1,-2,\ldots$)

and the generalized hyper geometric function, in a classical sense, has been defined [3] by

$${}_p\psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+\beta_1 n)\ldots\Gamma(\alpha_p+\beta_p n)\, z^n}{\Gamma(\rho_1+\mu_1 n)\ldots\Gamma(\rho_q+\mu_q n)\, n!},$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k z^k}{(b_1)_k \ldots (b_p)_k k!} \quad (p = q+1 \;\&\; |z|<1) \qquad (1.2)$$

and no denominator parameter equals zero or negative integer.

E. Wright [21] has further extended the generalization of the hypergeometric series in the following form.

$${}_p\psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+\beta_1 n)\ldots\Gamma(\alpha_p+\beta_p n)\, z^n}{\Gamma(\rho_1+\mu_1 n)\ldots\Gamma(\rho_q+\mu_q n)\, n!}, \qquad (1.3)$$

Where β_r and μ_t are real positive numbers such that

$$1+\sum_{t=1}^{q}\mu_t - \sum_{r=1}^{p}\beta_r > 0.$$

Where β_r and μ_t are equal to 1, Equation (1.3) differs from the generalized hyper geometric function pFq by a constant multiplier only. This generalized form of the hyper geometric function has been investigated by M. Dotsenko [2], V. Malovichko [8] others. One of the interesting special case considered in [2] has the following form.

$${}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+\frac{\omega}{\mu}n)}{\Gamma(c+\frac{\omega}{\mu}n)} \frac{z^n}{n!}; \qquad (1.4)$$

Where, ($R(a)>0; R(b)>0; R(c)>0$,

Here ω,μ both either positive or negative simultaneously, $|z|<1$.

The function ${}_2R_1^{\omega,\mu}(z)$ is not symmetric with respect to the parameters a and b . By letting $\frac{\omega}{\mu} = r > 0$

By equation (1.4),Virchenko et al.[18] defined the generalized hypergeometric function in a different sense as:

$${}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!} \qquad (1.5)$$

where $\tau > 0$, & $|z| < 1$.

If $\tau = 0$, then (1.5) reduces to a Gauss Hypergeometric Function ${}_2F_1(a,b;c;z)$,& $\tau > 0$ on $|z| < 1$., the function ${}_2R_1(a,b;c;\tau;z)$ is defined.

Rao et al. [13], [14] studied various properties of Generalized Hypergeometric Function in the light of fractional calculus.

The Riemann Liouville fractional integral of order υ is defined as [10];

$$I^{\upsilon} f(t) = \frac{1}{\Gamma\upsilon}\int_0^t (t-\xi)^{\upsilon-1} f(\xi)\,d\xi \qquad (1.6)$$

And the fractional differential operator of order μ defined as[10]

$$D^{\mu} f(t) = D^n\{I^{n-\mu} f(t)\}, \qquad (1.7)$$

Where $R(\mu) > 0$, and n is the smallest integer with the property that $n > R\mu$.

The Laplace transform of the function f(z) is defined as [1]:

$$L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z)\,dz. \qquad (1.8)$$

2. Fractional operators and the generalized hypergeometric function ${}_2R_1(a,b;c;\tau;z)$

Consider the function

$$f(t) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2} (t)^k = {}_1F_1(-n;1;t) \qquad (2.1)$$

Where $n \in C\,(R(n) > 0)$ and $|t| < 1$.

On the applying the fractional integral operators (1.6) of order υ on $f(t)$, which gives

$$I^{\nu} f(t) = \frac{1}{\Gamma(\nu)}\int_0^t [(t-\xi)^{\nu-1} f(\xi)]\,d\xi$$

$$= \frac{1}{\Gamma\nu}\int_0^t (t-\xi)^{\nu-1} \sum_{r=0}^{\infty} \frac{(-1)^r n!}{(n-r)!(r!)^2}(\xi)^r d\xi$$

$$= \frac{t^{\nu}}{\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(-n)_r}{(\nu+1)_r (r)!} t^r \qquad (2.2)$$

It can easily write this in form:

$$\frac{t^{\nu}}{\Gamma(\nu+1)}\, {}_1R_1(-n;\nu+1;1;t) = \frac{t^{\nu}}{\Gamma(\nu+1)}\, {}_1F_1(-n;\nu+1;t) \qquad (2.3)$$

We denote (2.3) as $R_t(1,\nu,-n)$ i.e.,

$$R_t(1,\nu,-n) = \frac{t^{\nu}}{\Gamma(\nu+1)}\, {}_1R_1(-n;\nu+1;1;t) = \frac{t^{\nu}}{\Gamma(\nu+1)}\, {}_1F_1(-n;\nu+1;t) \qquad (2.4)$$

Now applying the differential operator (1.7) of order μ on $f(t)$,we get

$$D^{\mu} f(t)=\left(\frac{d}{dt}\right)^{n}\left[I^{n-\mu}\sum_{r=0}^{\infty}\frac{(-1)^{r}n!}{(n-r)!\,(r!)^{2}}(t)^{r}\right] \quad (2.5)$$

Which yields

$$D^{\mu} f(t)=\frac{t^{-\mu}}{\Gamma(1-\mu)}\,{}_1R_1(-n;1-\mu;1;t)$$

$$=\frac{t^{-\mu}}{\Gamma(1-\mu)}\,{}_1F_1(-n;1-\mu;t) \quad (2.6)$$

on denoting (2.6) as

$$R_t(1,-\mu,-n)=\frac{t^{-\mu}}{\Gamma(1-\mu)}\,{}_1R_1(-n;1-\mu;1;t)$$

$$=\frac{t^{-\mu}}{\Gamma(1-\mu)}\,{}_1F_1(-n;1-\mu;t) \quad (2.7)$$

3. Properties of the function $R_t(1,v,-n)$ and $R_t(1,-\mu,-n)$

Theorem 3.1 If $n\in N$, then

$$I^{\lambda}R_t(1,v,-n)=R_t(1,\lambda+v,-n) \quad (3.1)$$

$$D^{\lambda}R_t(1,v,-n)=R_t(1,v-\lambda,-n) \quad (3.2)$$

The Laplace transform of $R_t(1,v,-n)$ is given as

$$L[\{R_t(1,v,-n)\}]=\frac{1}{s^{v+1}}\cdot y_n(1;-s)\} \quad (3.3)$$

Where $y_n(1;-s)$ is Generalized Bessel polynomial [7].

Proof: From equation (1.6) and equation (3.1), we get

$$I^{\lambda}R_t(1,v,-n)=\frac{1}{\Gamma(\lambda)}\int_0^t(t-\xi)^{\lambda-1}R_{\xi}(1,v,-n)d\xi$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^t(t-\xi)^{\lambda-1}\left(\frac{\xi^{v}}{\Gamma(v+1)}\,{}_1R_1(-n;v+1;1;t)\right)d\xi$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^t(t-\xi)^{\lambda-1}\left(\frac{\xi^{v}}{(v+1)}\sum_{r=0}^{\infty}\frac{(-n)_r\,\xi^{r}}{(v+1)_r r!}\right)d\xi \quad (3.4)$$

which gives us

$$I^{\lambda}R_t(1,v,-n)=\frac{1}{\Gamma(\lambda)}\int_0^1(1-x)^{\lambda-1}t^{\lambda-1}\left((xt)^{v}\sum_{r=0}^{\infty}\frac{(-n)_r(xt)^{r}}{(v+1)_r r!}\right)tdx$$

$$=\frac{t^{\lambda+v}}{\Gamma(\lambda+v+1)}\,{}_1R_1(-n;\lambda+v+1;1)=R_t(1,\ \lambda+v,-n) \quad (3.5)$$

This is the proof of equation (3.1).

$$D^{\lambda}R_t(1,v,-n)=D^{p}\{I^{p-\lambda}R_t(1,v,-n)\}=D^{p}[R_t(1,p-\lambda+v,-n)]$$

$$=D^{p}\left[\frac{t^{p-\lambda+v}}{\Gamma(p-\lambda+v+1)}\,{}_1R_1(-n;p-\lambda+v+1;1;t)\right]$$

$$=\frac{t^{v-\lambda}}{\Gamma(v-\lambda+1)}\,{}_1R_1(-n;v-\lambda+1;1;t)$$

$$=R_t(1,v-\lambda,-n) \quad (3.6)$$

Which is equation (3.2).

Now by taking Laplace transform of equation (2.4)

$$L[\{R_t(1,v,-n)\}]=L\left\{\frac{t^{v}}{\Gamma(v+1)}\,{}_1R_1(-n;v+1;1;t)\right\}$$

$$=\int_0^{\infty}e^{-st}\left\{\frac{t^{v}}{\Gamma(v+1)}\,{}_1R_1(-n;v+1;1;t)\right\}dt$$

$$=\frac{1}{s^{v+1}}\sum_{r=0}^{\infty}\frac{(-n)_r}{r!}\left(\frac{1}{s}\right)^{r}$$

$$=\frac{1}{s^{v+1}}\,{}_1F_0\left(-n;-;-\frac{1}{-s}\right)$$

$$=\frac{1}{s^{v+1}}y_n(1;-s)$$

Where $y_n(1;-s)$. is generalized Bessel Polynomial [7], this proves the equation (3.3)

Theorem 3.2 let $n\epsilon N$, then

$$I^{\lambda}R_t(1,-\mu,-n)=R_t(1,\lambda-\mu,-n), \quad (3.7)$$

$$D^{\lambda}R_t(1,-\mu,-n)=R_t(1,-\lambda-\mu,-n). \quad (3.8)$$

The laplace transform of $R_t(1,-\mu,-n)$ is given as

$$L\{R_t(1,-\mu,-n)\}=\frac{1}{s^{1-\mu}}y_n(1;-s) \quad (3.9)$$

Where $y_n(1,-s)$ is generalized Bessel polynomial [7].'

Proof: from equation (1.6) and equation (3.7),we get

$$I^{\lambda}R_t(1,-\mu,-n)=\frac{1}{\Gamma(\lambda)}\int_0^t(t-\xi)^{\lambda-1}R_{\xi}(1,-\mu,-n)d\xi$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^t(t-\xi)^{\lambda-1}\left(\frac{\xi^{-\mu}}{\Gamma(1-\mu)}\sum_{r=0}^{\infty}\frac{(-n)_r\xi^{r}}{(1-\mu)_r r!}\right)d\xi$$

Putting $\xi=xt$,we get

$$I^{\lambda}R_t(1,-\mu,-n)=\frac{1}{\Gamma(\lambda)}\int_0^1(1-x)^{\lambda-1}t^{\lambda-1}\left\{\frac{(xt)^{-\mu}}{\Gamma(1-\mu)}\sum_{r=0}^{\infty}\frac{(-n)_r(xt)^{r}}{(1-\mu)_r r!}\right\}t\,dx$$

$$=\frac{t^{\lambda-\mu}}{\Gamma(\lambda-\mu+1)}\,{}_1R_1(-n:\lambda-\mu+1;1;t)=R_t(1,\lambda-\mu,-n)$$

Which proves the equation (3.7)

Now from equation (1.7) and equation (3.8)

$$D^{\lambda}(R_t(1,-\mu,-n))=D^{p}[I^{p-\lambda}R_t(1,-\mu,-n)]$$

$$=D^{p}[R_t(1,p-\lambda-\mu,-n)]$$

$$=D^{p}\frac{t^{p-\lambda-\mu}}{\Gamma(p-\lambda-\mu+1)}\,{}_1R_1(-n;p-\lambda-\mu+1;1;t)$$

$$=\frac{t^{-\lambda-\mu}}{\Gamma(-\lambda-\mu+1)}\,{}_1R_1(-n;-\lambda-\mu+1;1;t)$$

$$=R_t(1,-\lambda-\mu,-n)$$

This is proof of equation (3.8).

Now taking Laplace transform of (2.7), it yields

$$L[R_t(1,-\mu,-n)]=L\left[\left\{\frac{t^{-\mu}}{\Gamma(1-\mu)}\,{}_1R_1(-n;1-\mu;1;t)\right\}\right]$$

$$=\int_0^{\infty}e^{-st}\left\{\frac{t^{-\mu}}{\Gamma(1-\mu)}\sum_{r=0}^{\infty}\frac{(-n)_r(t)^{r}}{(1-\mu)_r r!}\right\}dt$$

$$=\frac{1}{s^{1-\mu}}\sum_{r=0}^{\infty}\frac{(-s)_r}{r!s^{r}}$$

$$=\frac{1}{s^{1-\mu}}\,{}_1F_0\left(-n;-;-\frac{1}{-s}\right)$$

$$=\frac{1}{s^{1-\mu}}y_n(1;-s)$$

Thus the Laplace Transform of $R_t(1,-\mu,-n)$ is given as

$$L[\{R_t(1,-\mu,-n)\}]=\frac{1}{s^{1-\mu}}y_n(1;-s)$$

Where $y_n(1;-s)$ is Generalized Bessel polynomial [7]

This is the proof of equation (3.9).

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