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INTRODUCTORY NOTE ON PROJECTIVE TRANSFORMATION

Mathematics

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ABSTRACT

Every branch of natural sciences, social sciences and formal sciences deals with specific sets of elements and the structure of relationship between these elements. Each science studies some definite types of transformations and specific invariants relevant to these transformations.. Mathematics provides models having same structure those in social, physical and biological sciences. It also deals with sets, structures on these sets, transformations in them and their invariants.

KEYWORDS

Fractional, transformation, model, projective, affine

Mathematics deals with relatively more abstract sets and structures than other sciences. Mathematics essentially deals with three basic or mother structures, viz., order, algebraic, topological, and most mathematical systems exhibit these structures singly or fused together. The central idea in modern mathematics is that the study of a structure is best made in terms of transformations which preserve that structure [2] .A physical, biological or social situation may be highly complicated, but at one time we are interested only in some particular aspect of it. We first find an appropriate transformation to convert this situation into a mathematical form which preserves the structure we are interested in. This process is called mathematical model making. The mathematical model is as good or as bad as its capacity to preserve the particular aspect of the structure we are considering. So our problem is to find a transformation which is as close to reality as possible and which is capable of being handled by appropriate simplifying mathematical transformations. Most of the modern curriculum reform projects in school mathematics [3] have given an important place to transformation geometry, especially to the study of isometric (distance preserving) transformations. Several new undergraduate courses also contain a discussion of metric geometry over affine space [4, 5].

An affine transformation is given by[1]

$$x' = ax + by + d,\ y' = lx + my + n,\ am - bl \neq 0 \text{..........(1)}$$

Here $am - bl = \begin{vmatrix} a & b \\ l & m \end{vmatrix}$ is known as the determinant of the transformation.

(1) is a one –one mapping from the xy plane onto itself. Solving (1) for x and y, we get

$$x = \frac{m}{am-bl}x' - \frac{b}{am-bl}y' + \frac{bn-dm}{am-bl},\ \text{and}\ y = -\frac{l}{am-bl}x' + \frac{a}{am-bl}y' + \frac{dl-an}{am-bl} \text{....(2)}$$

Thus, to a given point$(x',\ y')$, there corresponds a unique point (x, y) if $am - bl \neq 0$ ... ... (3)

$$\text{Also } \left(\frac{m}{am-bl}\right)\left(\frac{a}{am-bl}\right) - \left(-\frac{b}{am-bl}\right)\left(-\frac{l}{am-bl}\right) \neq 0$$

Thus Equations (ii) also determine an affine transformation.

A Projective transformation is transformation which has following three properties [6]-

i. It sends plane onto itself in a one-one way except that the points on one line lying in plane have no images and the points on the other line may have no pre images.
ii. It preserves co linearity of all points not line on vanishing line.
iii. It preserves the cross ratio of four points not line on a vanishing line.

Now we show that every fractional transformation with some condition always becomes projective transformation. Also some important properties of projective transformation will be discussed in this paper.

Fractional Transformation is given by [6]

$$x' = \frac{\alpha_1 x + \alpha_2 y + \alpha_3}{\gamma_1 x + \gamma_2 y + \gamma_3}, y' = \frac{\beta_1 x + \beta_2 y + \beta_3}{\gamma_1 x + \gamma_2 y + \gamma_3} \text{.....(4)}$$

Here $\gamma_1, \gamma_2, \gamma_3$ are not all zero.

Determinant of the transformation $\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0$ .....(5)

$\therefore \Delta = \alpha_3 A_3 + \beta_3 B_3 + \gamma_3 C_3 \neq 0$, where A_3, B_3, C_3 are cofactors of $\alpha_3, \beta_3, \gamma_3$ respectively in Δ.

From (1) $x'(\gamma_1 x + \gamma_2 y + \gamma_3) = \alpha_1 x + \alpha_2 y + \alpha_3$ & $y'(\gamma_1 x + \gamma_2 y + \gamma_3) = \beta_1 x + \beta_2 y + \beta_3$

$\therefore (x'\gamma_1 - \alpha_1)x + (x'\gamma_2 - \alpha_2)y = \alpha_3 - x'\gamma_3$ ....(6)

& $(y'\gamma_1 - \beta_1)x + (y'\gamma_2 - \beta_2)y = \beta_3 - y'\gamma_3$........(7)

Solving by Cramer's Rule, we get

$$x = \frac{\begin{vmatrix} \alpha_3 - x'\gamma_3 & x'\gamma_2 - \alpha_2 \\ \beta_3 - y'\gamma_3 & y'\gamma_2 - \beta_2 \end{vmatrix}}{\begin{vmatrix} x'\gamma_1 - \alpha_1 & x'\gamma_2 - \alpha_2 \\ y'\gamma_1 - \beta_1 & y'\gamma_2 - \beta_2 \end{vmatrix}} = \frac{(\alpha_3 - x'\gamma_3)(y'\gamma_2 - \beta_2) - (\beta_3 - y'\gamma_3)(x'\gamma_2 - \alpha_2)}{(x'\gamma_1 - \alpha_1)(y'\gamma_2 - \beta_2) - (x'\gamma_2 - \alpha_2)(y'\gamma_1 - \beta_1)}$$

$$= \frac{(\beta_2\gamma_3 - \beta_3\gamma_2)x' + (\alpha_3\gamma_2 - \alpha_2\gamma_3)y' + (\alpha_2\beta_3 - \alpha_3\beta_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)x' + (\alpha_2\gamma_1 - \alpha_1\gamma_2)y' + (\alpha_1\beta_2 - \alpha_2\beta_1)}$$

$$\left.\begin{aligned} x &= \frac{A_1 x' + B_1 y' + C_1}{A_3 x' + B_3 y' + C_3} \\ \& \ y &= \frac{A_2 x' + B_2 y' + C_2}{A_3 x' + B_3 y' + C_3} \end{aligned}\right\} \text{......... (8)}$$

Where $A_1, B_1, C_1; A_2, B_2, C_2;\ A_3, B_3, C_3$ are the cofactors of $\alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2; \alpha_3, \beta_3, \gamma_3$ respectively in Δ.

For solution (8) and (9) to be valid A_3, B_3, C_3 all should not vanish.

$\because \Delta \neq 0$; A_3, B_3, C_3 cannot all vanish.

From (4) we see that every point on the xy plane except lying on line

$\gamma_1 x + \gamma_2 y + \gamma_3 = 0$......(9) has a unique image and points on this line have no image.

Similarly from (8), we see that every point of the x'y' plane except points lying on line $A_3 x' + B_3 y' + C_3 = 0$ -------------(10) has a unique pre image, and the points on this line have no pre image. So lines $\gamma_1 x + \gamma_2 y + \gamma_3 = 0$ and $A_3 x' + B_3 y' + C_3 = 0$ are called vanishing lines of the plane xy and x'y' respectively.

From (8) we find a fractional transformation inverse to (4) if its determinant

$$\delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0 \text{ ....(11)}$$

$$\text{We have } \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 & \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 & \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3 \\ \beta_1 A_1 + \beta_2 A_2 + \beta_3 A_3 & \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 & \beta_1 C_1 + \beta_2 C_2 + \beta_3 C_3 \\ \gamma_1 A_1 + \gamma_2 A_2 + \gamma_3 A_3 & \gamma_1 B_1 + \gamma_2 B_2 + \gamma_3 B_3 & \gamma_1 C_1 + \gamma_2 C_2 + \gamma_3 C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}$$

$$\therefore \Delta\delta = \Delta^3 \Longrightarrow \delta = \Delta^2$$

$\Delta \neq 0 \Rightarrow \delta \neq 0 \Longrightarrow$ Transformation given by (8) is fractional transformation inverse to transformation defined by (4) .

Cases-

(i) Let four collinear points A(a,0), B(b,0), C(c,0),D(d,0) be on the x axis and let their images be A', B', C', D'. Then coordinates of A', B', C', D' are

$$A'\left(\frac{\alpha_1 a + \alpha_3}{\gamma_1 a + \gamma_3}, \frac{\beta_1 a + \beta_3}{\gamma_1 a + \gamma_3}\right), B'\left(\frac{\alpha_1 b + \alpha_3}{\gamma_1 b + \gamma_3}, \frac{\beta_1 b + \beta_3}{\gamma_1 b + \gamma_3}\right), C'\left(\frac{\alpha_1 c + \alpha_3}{\gamma_1 c + \gamma_3}, \frac{\beta_1 c + \beta_3}{\gamma_1 c + \gamma_3}\right), D'\left(\frac{\alpha_1 d + \alpha_3}{\gamma_1 d + \gamma_3}, \frac{\beta_1 d + \beta_3}{\gamma_1 d + \gamma_3}\right)$$

$$(A'B')^2 = \left(\frac{\alpha_1 b+\alpha_3}{\gamma_1 b+\gamma_3} - \frac{\alpha_1 a+\alpha_3}{\gamma_1 a+\gamma_3}\right)^2 + \left(\frac{\beta_1 b+\beta_3}{\gamma_1 b+\gamma_3} - \frac{\beta_1 a+\beta_3}{\gamma_1 a+\gamma_3}\right)^2$$

$$= \left(\frac{(\alpha_1 b+\alpha_3)(\gamma_1 a+\gamma_3) - (\alpha_1 a+\alpha_3)(\gamma_1 b+\gamma_3)}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}\right)^2 + \left(\frac{(\beta_1 b+\beta_3)(\gamma_1 a+\gamma_3) - (\beta_1 a+\beta_3)(\gamma_1 b+\gamma_3)}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}\right)^2$$

$$= \left(\frac{\alpha_3\gamma_1 a+\gamma_3\alpha_1 b-\alpha_3\gamma_1 b-\alpha_1\gamma_3 a}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}\right)^2 + \left(\frac{\beta_3\gamma_1 a+\beta_1\gamma_3 b-\beta_3\gamma_1 b-\beta_1\gamma_3 a}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}\right)^2$$

$$= \left(\frac{\alpha_3\gamma_1(a-b)-\alpha_1\gamma_3(a-b)}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}\right)^2 + \left(\frac{\beta_3\gamma_1(a-b)-\beta_1\gamma_3(a-b)}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}\right)^2$$

$$= (a-b)^2 \frac{[(\alpha_3\gamma_1-\alpha_1\gamma_3)^2+(\beta_3\gamma_1-\beta_1\gamma_3)^2]}{(\gamma_1 b+\gamma_3)^2(\gamma_1 a+\gamma_3)^2}$$

$$\therefore A'B' = \frac{\sqrt{(\alpha_3\gamma_1-\alpha_1\gamma_3)^2+(\beta_3\gamma_1-\beta_1\gamma_3)^2}}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}(a-b)$$

$$\Rightarrow A'B' = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}(a-b) = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}AB$$

Where $K = \sqrt{(\alpha_3\gamma_1-\alpha_1\gamma_3)^2+(\beta_3\gamma_1-\beta_1\gamma_3)^2}$

Similarly

$$A'C' = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}(a-c) = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}AC;$$

$$A'D' = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}(a-d) = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}AD;$$

$$B'C' = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}(b-c) = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}BC;$$

$$B'D' = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}(b-d) = \frac{K}{(\gamma_1 b+\gamma_3)(\gamma_1 a+\gamma_3)}BD;$$

Now

$$\frac{A'C'}{B'C'}.\frac{B'D'}{A'D'} = \frac{(a-c)}{(b-c)}.\frac{(b-d)}{(a-d)} = \frac{AC}{BC}.\frac{BD}{AD}$$

Cross ratio (A', B', C', D') = Cross ratio (A, B, C, D)

Or $(A'\ B'\ C'\ D') = (A, B, C, D)$

So we can say that fractional transformation (4) preserves the cross ratio of any four collinear points which do not lie on vanishing line $\gamma_1 x+\gamma_2 y+\gamma_3 = 0$

Also we can say that in general fractional transformation (4) do not preserve distances between points.

(ii) Let $lx+my+n=0$ ……….(12) where l, m both are non zero, be any straight line different from $\gamma_1 x+\gamma_2 y+\gamma_3 = 0$

Its image by (4) is

$$l\left[\frac{A_1x'+B_1y'+C_1}{A_3x'+B_3y'+C_3}\right] + m\left[\frac{A_2x'+B_2y'+C_2}{A_3x'+B_3y'+C_3}\right] + n = 0$$

Or $(lA_1+mA_2+nA_3)x' + (lB_1+mB_2+nB_3)y' + (lC_1+mC_2+nC_3) = 0$

………(13)

$$\text{If} \quad \left.\begin{aligned} lA_1+mA_2+nA_3 &= 0 \\ lB_1+mB_2+nB_3 &= 0 \\ lC_1+mC_2+nC_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots(14)$$

Now $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \delta \neq 0$

By Cramer's rule equation (14) gives $l = m = n = 0$, which is contradiction.

Thus when the coefficients of both x and y in (14) are not zero, the coefficients of x', y' in (13) cannot be zero.

Let P_1 be the point of intersection of line (12) and (9) and P_2 be the point of intersection of line (13) and (10) .

Then point P_1 has no image and P_2 has no pre image.

Thus every point except P_1 on line (12) has image line (13) and every point on line (13) except point P_2 has a unique pre image on line (12).

∴ Punctured line (12) is mapped on punctured line (13).

When line (12) is parallel to vanishing line $\gamma_1 x+\gamma_2 y+\gamma_3 = 0$ then complete line (12) is mapped onto the complete line (13) parallel to line $A_3x'+B_3y'+C_3 = 0$.

If there are three collinear points P_1, P_2, P_3 none of which lies on the line $\gamma_1 x+\gamma_2 y+\gamma_3 = 0$, then their images P_1', P_2', P_3' would also be collinear and none of them would lie on line $A_3x'+B_3y'+C_3 = 0$.

So we can say that fractional transformation of the form (4) with condition (5) gives a projective transformation.

Results-

1. When $\gamma_1 = \gamma_2 = 0$

Then $A_3 = 0, B_3 = 0$

Then every point of the plane has unique image and unique pre image.

Also $\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{vmatrix} = \gamma_3(\alpha_1\beta_2-\alpha_2\beta_1) = \gamma_3 C_3 \neq 0 \Rightarrow$ both γ_3 and C_3 are not zero

In this case

$$\left.\begin{aligned} x' &= \frac{\alpha_1 x+\alpha_2 y+\alpha_3}{\gamma_3} = \frac{\alpha_1}{\gamma_3}x+\frac{\alpha_2}{\gamma_3}y+\frac{\alpha_3}{\gamma_3} \\ \&\ y' &= \frac{\beta_1 x+\beta_2 y+\beta_3}{\gamma_3} = \frac{\beta_1}{\gamma_3}x+\frac{\beta_2}{\gamma_3}y+\frac{\beta_3}{\gamma_3} \ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Here $\frac{\alpha_1}{\gamma_3}.\frac{\beta_2}{\gamma_3} - \frac{\alpha_2}{\gamma_3}.\frac{\beta_1}{\gamma_3} = \frac{\alpha_1\beta_2-\alpha_2\beta_1}{\gamma_3^2} = \frac{C_3}{\gamma_3^2} \neq 0$

Also from (8)

$$\left.\begin{aligned} x &= \frac{A_1x'+B_1y'+C_1}{C_3} = \frac{A_1}{C_3}x'+\frac{B_1}{C_3}y'+\frac{C_1}{C_3} \\ y &= \frac{A_2x'+B_2y'+C_2}{C_3} = \frac{A_2}{C_3}x'+\frac{B_2}{C_3}y'+\frac{C_2}{C_3} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Its determinant is $\frac{A_1}{C_3}\frac{B_2}{C_3} - \frac{A_2}{C_3}\frac{B_1}{C_3} = \frac{A_1B_2-A_2B_1}{C_3^2} = \frac{\gamma_3\Delta}{C_3^2} \neq 0$

∴ Both transformations (15) and (16) are inverse to each other and represent affine transformations.

Thus in this case projective transformation becomes affine transformation as defined in (1).

Therefore the set of all affine transformation is a proper subset of all projective transformation.

An affine transformation maps straight lines onto straight lines. A straight line maps a family of parallel straight lines onto another family of parallel straight lines. In general affine transformations do not preserve distances between points. Affine transformations which preserve distances are called isometric transformations.

2. Projective transformation (4) transforms an equation of second degree into an equation of second degree. We can say projective transformation map conics onto conics.

General equation of any conic is

$ax^2+2hxy+by^2+2gx+2fy+c=0$

We can write $[x\ \ y\ \ 1]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

Using (8), its image is given by

$$[x'\ \ y'\ \ 1]\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0$$

which is conic of the form $a'x'^2+2h'x'y'+b'y'^2+2g'x'+2f'y'+c'=0$

Matrices' of image conic and original conic are related by

$$\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \delta^2\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Delta^2\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

For projective transformation $\Delta \neq 0$, and for a non degenerate conic $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$

$$\rightarrow \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} \neq 0$$

∴A projective transformation maps a degenerate conic onto a degenerate conic and a non degenerate conic onto a non degenerate conic.

Also we see that a', h', b' depends on a, b, c, h, g, f etc.

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∴ Sign of $a'b' - h'^2$ and sign of $ab - h^2$ will not always be same. Therefore a projective transformation can map a non degenerate conic of one type onto a non degenerate conic of another type.

3. A conic can meet the vanishing line $\gamma_1 x + \gamma_2 y + \gamma_3 = 0$ in 0,1 or 2 points according as it does not intersect, touch or cut the conic in two distinct points. Points on vanishing line have no images. Therefore a conic punctured in 0,1 or 2 points will be mapped onto a conic punctured in 0,1, or 2 points respectively.
4. In general projective transformation does not preserve distances between points.
5. There is a unique projective transformation that maps four given points, no three of which are collinear, onto four points, no three of which are collinear.
6. There is a unique projective transformation in which a given line has no image and which maps three non-collinear points onto three non-collinear points.

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