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NUMERICAL COMPARISON OF TWO-LEVEL CRANK-NICOLSON AND THREE-LEVEL MODIFIED DU-FORT-FRANKEL METHODS FOR THE ONE-DIMENSIONAL HEAT EQUATION

Mathematics

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ABSTRACT

A three-level implicit finite difference scheme similar to the ordinary Crank–Nicolson (CN) method was developed in this study. The proposed method, referred to as the Modified Crank–Nicolson Du-Fort and Frankel (MCN-DF) scheme, was applied to the numerical solution of the one-dimensional heat equation. Numerical results obtained using the ordinary CN and MCN-DF methods were compared. The findings of the study indicate that the three-level MCN-DF scheme provides improved numerical accuracy while retaining the stability characteristics associated with implicit finite difference methods. The proposed method therefore offers an effective alternative approach for solving one-dimensional heat transfer problems governed by parabolic partial differential equations. The study contributes to the development of higher-accuracy finite difference techniques for the numerical solution of heat equations and related physical models.

KEYWORDS

INTRODUCTION

Heat equation is an elementary parabolic equation, a model representing flow of heat under normal circumstances. Many scientists in the past have had interest on heat equation. It has thus both analytic and numerical solutions with varied levels of accuracy. [1], [2], [3], [4], [5], [6], [7], [8], [9] and [10] gives the analytical solution of the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \ (0 \leq x \leq 1) \times (t \geq 0) \qquad (1)$$

$$u(0,t) = u(1,t) = 0 \qquad (2)$$

$$\text{with } u(x,t) = e - \pi^2 t sin\pi x \qquad (3)$$

The authors [1], [3], [5], [7], [8] and [10] have also discussed finite difference solution of this problem. In particular they have discussed the Schmidt and the two-level Crank-Nicolson method. Author [7] goes further to discuss the Douglas method.

In this paper a three-level Modified Crank-Nicolson Du-Fort and Frankel method for solving problem (1) is developed. The work is an improvement of the contribution in [11] where the Modified Crank-Nicolson was introduced.

The objective of the research is to: develop Hybrid Modified Crank-Nicolson Du-Fort and Frankel (MCN-DF) method for the solution of 1-D heat equation; compare the numerical solution of Hybrid Modified Crank-Nicolson Du-Fort and Frankel (MCN-DF), Modified Crank-Nicolson (MCN) and ordinary Crank -Nicolson (CN) and finally to analyze the stability of the scheme developed.

The paper is organized as follows: in section 2, the 3-level modified Crank-Nicolson Du-Fort scheme for solving the heat equation (1)-(2) is developed. In section 3, results from both ordinary Crank-Nicolson and the modified Crank-Nicolson Du-Fort and Frankel methods are displayed and discussed. Section 4 gives the conclusion of the research and section 5 give the recommendations.

DEVELOPMENT OF THE FINITE DIFFERENCE SCHEMES

Modified Crank-Nicolson Scheme

We consider the problem (1). The ordinary CN discretization is

$$\Delta_t U_{m,n} = \frac{\propto}{2h^2} \delta_x^2 (U_{m,n} + U_{m,n+1}) \qquad (4)$$

which is a two-level scheme. The modified Crank-Nicolson method (MCN) which is a three-level scheme is obtained as follows:

$$\Delta_t U_{m,n} = \frac{\propto}{3h^2} \delta_x^2 (U_{m,n-1} + U_{m,n} + U_{m,n+1}) \qquad (5)$$

The MCN utilize one extra grid point at the lower level of the point in question. Equation (4) and (5) Δ_t and δ_x^2 are the forward time difference operator and second order central space difference respectively. Equation yields:

$$-\frac{k\alpha}{h^2} U_{m-1,n+1} + \left(3 + 2\frac{k\alpha}{h^2}\right) U_{m,n+1} - \frac{k\alpha}{h^2} U_{m+1,n+1} = \frac{k\alpha}{h^2} U_{m-1,n-1} - 2\frac{k\alpha}{h^2} U_{m,n-1} + \frac{k\alpha}{h^2} U_{m+1,n-1} + \frac{k\alpha}{h^2} U_{m-1,n} + \left(3 - 2\frac{k\alpha}{h^2}\right) U_{m,n} + \frac{k\alpha}{h^2} U_{m+1,n} \qquad (6)$$

Hybrid Modified Crank-Nicolson Du-Fort and Frankel (MCN-DF) Scheme The Modified Crank-Nicolson Du-Fort and Frankel is obtained by approximating the left hand side of equation (1) by $\mu \delta_t U_{m,n}$ and replacing the term $U_{m,n}$ in the right hand side of equation (6) by $\frac{1}{2}(U_{m,n-1} + U_{m,n+1})$. Equation yields:

$$-\frac{2}{3}\frac{k\alpha}{h^2} U_{m-1,n+1} + \left(1 + \frac{4}{3}\frac{k\alpha}{h^2}\right) U_{m,n+1} - \frac{2}{3}\frac{k\alpha}{h^2} U_{m+1,n+1} = \frac{2k\alpha}{3h^2} U_{m-1,n} - \frac{4}{3}\frac{k\alpha}{h^2} U_{m,n} + \frac{2}{3}\frac{k\alpha}{h^2} U_{m+1,n} + \frac{2}{3}\frac{k\alpha}{h^2} U_{m-1,n-1} + \left(1 - \frac{4}{3}\frac{k\alpha}{h^2}\right) U_{m,n-1} + \frac{2}{3}\frac{k\alpha}{h^2} U_{m+1,n-1} \qquad (7)$$

PRESENTATION OF RESULTS

In this section results (solution) of the two schemes are presented and discussed. The following data: h=0.1, k=0.001 and a=1.0 is used in all cases. A table of solutions and absolute errors is presented also 2-D (graphical) and 3-D solutions are presented. Two- dimensional solutions from the two schemes are as shown below.

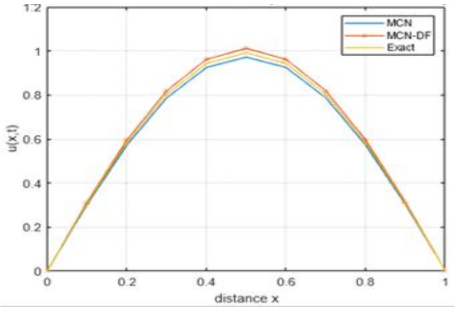

Figure 1: Solutions for the 1-D heat equations at t=0.001

Clearly the solutions obtained using CN, MCN, MCN-DF and that of exact coincide. This means that these methods are fairly accurate. A table of absolute errors in solutions is provided below to determine which of the CN, MCN and MCN-DF methods is more accurate.

Table 1: solutions of the 1-D heat equation at t=0.01 from the three schemes

x	Exact	CN	MCN-DF	MCN
0	0	0	0	0
0.1	0.29389243	0.31208132	0.296807196	0.28020137
0.2	0.55901662	0.59360305	0.564560835	0.532974678
0.3	0.76942037	0.8170395	0.777051326	0.733576711
0.4	0.90450789	0.96048754	0.91347862	0.862371144

0.5	0.95105588	1.00991637	0.960488262	0.906750682
0.6	0.90450789	0.96048754	0.91347862	0.862371144
0.7	0.76942037	0.8170395	0.777051326	0.733576711
0.8	0.55909309	0.59360305	0.564560835	0.53297467
0.9	0.29389243	0.31208132	0.296807196	0.28020137
1	0	0	0	0

Table 2: Absolute errors in solution from the two methods of 1-D heat equation

x	CN	MCN-DF	MCN
0	0	0	0
0.1	0.01818889	0.002914766	-0.01369106
0.2	0.03458643	0.005544215	-0.026041942
0.3	0.04761913	0.007630956	-0.035843659
0.4	0.05597965	0.00897073	-0.042136746
0.5	0.05886049	0.009432382	-0.044305198
0.6	0.05597965	0.00897073	-0.042136746
0.7	0.04761913	0.007630956	-0.035843659
0.8	0.03450996	0.005467745	-0.02611842
0.9	0.01818889	0.002914766	-0.01369106
0	0	0	0

It can be seen from **Table 2** that absolute errors for the MCN-DF is the least, meaning that it is the most accurate.

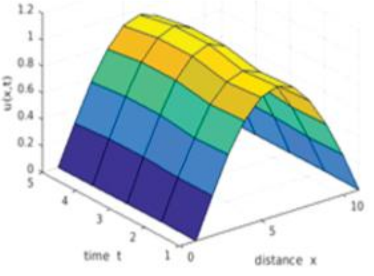

Figure 2: MCN Solution for the 1-D Heat Equation

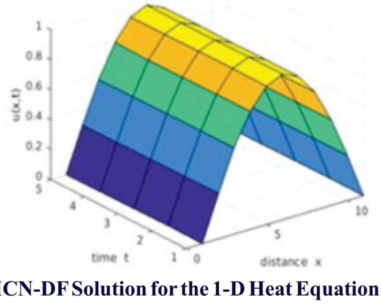

Figure 3: MCN-DF Solution for the 1-D Heat Equation

Significantly, it is observed that 3-D for the two methods have solutions of the same shape. This is typical of heat distribution from a source through a medium of uniform density in one direction.

CONCLUSION

In the research:

- Hybrid Modified Crank-Nicolson Du-Fort and Frankel (MCN-DF) method for the solution of 1-D heat equation was successfully developed.
- It is determined that the 3-level MCN-DF Numerical method produces more accurate results compared to the 2-level ordinary Crank-Nicolson Numerical method. All the methods developed above are implicit and therefore are unconditionally stable.

Recommendation

The Hybrid scheme can be developed for the solution of other 2D and 3D parabolic equations

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