Research Paper Mathematics

MHD FLOWS-IMAGE OF STREAMLINES IN BELTRAMI SURFACE

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ABSTRACT

Let S be a surface. $\bar{S}$ the corresponding Beltrami surface obtained by surface revolution of S. By using the fundamental magnitudes and the flow on the surface S we study flow on the surface $\bar{S}$ and it is proved that the streamlines in the Beltrami surface $\bar{S}$ are concentric circles or radial.

KEYWORDS | Beltrami surface, Image streamlines, Fundamental magnitudes, concentric circles and radial

1.1 Introduction

The Navier-Stokes equations are highly non linear partial differential equations in almost every real situation. Because of this, most problems are difficult or impossible to solve. Therefore, some special methods which can handle the above type of equations have gained considerable importance. One such technique is the Differential Geometry technique .The geometric technique has attracted the investigator in fluid flow theories as it not only provides the information of flow structure, but also accommodates quite a good number of analytical flows.

Exploiting the geometric properties of streamline triad, first time Kanwal [4] has attempted to describe the equations governing spatial flows in intrinsic form. Truesdell [8] has surveyed the literature and attempted to describe the role of geometries in spatial gas flows. Nemenyi [5] studied the geometric properties of plane gas flows, utilizing the orthogonal geometric net related to the streamline and their orthogonal trajectories.

Chandna, M.R.Garg [2] have considered plane steady, aligned magneto hydrodynamic flow and they have proved that the streamlines are concentric circles , when (i) they are straight lines but not parallel, (ii) they are involute of a curve C. Recently C.S.Bagewadi and K.N. Prasannakumar [1] have studied the geometry of stream lines in spherical, inverse and parallel

surfaces as image surfaces. They have considered non parallel straight streamlines enveloping a curve C in plane steady aligned MGD flow on the surface S. By taking tangent lines to C and their orthogonal trajectories (the involutes of C) as a system of orthogonal curvilinear co-ordinates they have established that the streamlines are concentric circles and the magnetic permeability of the fluid medium is constant. Siddabasappa etal [6] established that in a Beltrami surface the image streamlines of steady MGD flow are concentric circles and the magnetic permeability acts in the tangential direction to Beltrami surface.

In this paper by using fundamental magnitudes it is established that If the streamlines in steady plane magnetohydrodynamic flow of a viscous incompressible fluid of infinite electrical conductivity in the surface S are straight lines but not parallel and envelope a curve C, the tangent lines to C i.e., η = constant and the involutes of the curve C i.e., ξ = constant are taken as orthogonal curvilinear co-ordinates, then the streamlines in the Beltrami surface $\overline{\boldsymbol{S}}$ are concentric circles or radial .

1.2 SURFACES

Let $r: D \to E^3$ be a differential map defined by$(\xi,\eta) \to r(\xi,\eta)$, where D is an open subset of E^2 and $(\xi,\eta)\epsilon D$. Then the curve $\xi \to r(\xi,\eta_0)$ defined by $\eta = \eta_0$ is called ξ-parameter curve and the curve $\eta \to r(\xi_0,\eta)$ defined by $\xi = \xi_0$ is called η-parameter curve. By giving values for ξ and η we get points on the surface. The intersection of the curves $\xi = \xi_0$ and $\eta = \eta_0$ represents a point (ξ_0,η_0) on the surface. The pairs (ξ,η) define the curvilinear co-ordinates on the surface. The surface is covered by these two families of curves. If these two curves intersect orthogonally then they are said to form orthogonal curvilinear co-ordinate system, otherwise non-orthogonal.

The vectors $\frac{\partial r}{\partial \xi}$ and$\frac{\partial r}{\partial \eta}$ are called Tangent vectors to curves $\eta = \eta_0$ and $\xi = \xi_0$ at the points (ξ_0 , η_0) respectively. Set

$$r_1 = \frac{\partial r}{\partial \xi}, r_2 = \frac{\partial r}{\partial \eta}, r_{12} = \frac{\partial^2 r}{\partial \xi \partial \eta} = r_{21} \quad \text{and} \quad r_{11} = \frac{\partial^2 r}{\partial \xi^2}, r_{22} = \frac{\partial^2 r}{\partial \eta^2}$$

The arc length dS in curvilinear co-ordinate system (ξ,η) is given by

$$dS^2 = dr.dr = Ed\xi^2 + 2Fd\xi d\eta + Gd\eta^2 \qquad \text{(a)}$$

where E, F and G are called first fundamental magnitudes and are given by

$E = r_1^2 = g_1^2(\xi,\eta),\quad G = r_2^2 = g_2^2(\xi,\eta)\quad , F = r_2.r_2 = |r_1||r_2|cos\theta = \sqrt{EG}\,cos\theta$

$n = \frac{r_{1x}r_2}{W}$ is the unit normal to the surface and $W^2 = EG - F^2$ .Let K denote the Gaussian curvature, then

$$K = \frac{LN-M^2}{W^2} = \frac{1}{2W}\left[\left\{\frac{F}{EW}\frac{\partial E}{\partial \eta} - \frac{1}{W}\frac{\partial G}{\partial \eta}\right\} + \left\{\frac{2}{W}\frac{\partial F}{\partial \xi} - \frac{1}{W}\frac{\partial E}{\partial \eta} - \frac{F}{EW}\right\}\right]$$

is called Gauss characteristic equation. If e_1 and e_2 are unit tangent vectors defined on a open subset D of E^2 and $V = (V_1.V_2)$ is a vector, then in an orthogonal curvilinear co-ordinate system, we have-

$$grad\ \Phi = \nabla\Phi = \frac{1}{g_1}\frac{\partial \Phi}{\partial \xi}e_1 + \frac{1}{g_2}\frac{\partial \Phi}{\partial \eta}e_2 \qquad (1)$$

$$div\ \vec{V}\ = \nabla.\vec{V} = \frac{1}{g_1 g_2}\left\{\frac{\partial(V_1 g_2)}{\partial \xi} + \frac{\partial(V_2 g_1)}{\partial \eta}\right\}$$

$$curl\ \vec{V} = \nabla \text{ x } \vec{V} = \frac{1}{g_1 g_2}\left[\frac{\partial(V_2 g_2)}{\partial \xi} - \frac{\partial(V_1 g_1)}{\partial \eta}\right]e_3$$

Where e_3 is a unit vector orthogonal to both e_1 and e_2.

1.3 Beltrami surface

Let Φ be a scalar point function in the (ξ,η) plane. It is known that the gradient of Φ at any point P of the plane is a vector quantity. Let r be the position vector of any point in the (ξ,η) plane. Construct a vector field **V** from Φ as follows:

$$V = (G\Phi_1 - F\Phi_2)r_1 + (E\Phi_2 - F\Phi_1)r_2 \qquad (2)$$

where $\Phi_1 = \frac{\partial \Phi}{\partial \xi}$, $\Phi_2 = \frac{\partial \Phi}{\partial \eta}$.

Let Ψ be another scalar point function in the (ξ,η) plane. Construct a vorticity vector Ω from the scalar point functions Φ and Ψ as $\Omega = \Psi\nabla\Phi$

In this case the surface generated by Φ= constant is called as Beltrami surface and Ψ= constant is called distance function surface for the family of Beltrami surfaces. The following identities hold true –

$(\nabla\Phi)^2 = W^{-2}(E\Phi_2^2 - 2F\Phi_1\Phi_2 + G\Phi_1^2)$, $\nabla\Phi \text{ x } \nabla\Psi = W^{-1}(\Phi_1\Psi_2 - \Phi_2\Psi_1)n$

$$\nabla\Phi.\nabla\Psi = W^{-2}[E\Phi_2\Psi_2 - F(\Phi_1\Psi_2 - \Phi_2\Psi_1) + G\Phi_1\Psi_1] \quad (3)$$

where n is a unit vector to the $\xi - \eta$ plane. The curves Ψ= constant are orthogonal trajectories of the curves Φ= constant if the gradients of the two functions are everywhere perpendicular and the condition of the orthogonality of the two systems of curves is$\nabla\Phi.\nabla\Psi = 0$.

If Δ_1 is the Beltrami differential operator of the first order then $\Delta_1\Phi$ is the square of the magnitude of $\nabla\Phi$ that is – $\Delta_1\Phi = (\nabla\Phi)^2$

The mixed differential operator of the first order is the scalar whose product of the gradients of Φ andΨ , that is – $\Delta_1(\Phi,\Psi) = \nabla\Phi.\nabla\Psi$

Let E, F, G be the first fundamental magnitudes for the parametric variable functions Φ and Ψ in the Beltrami surface and are given by –

$$W^{-2} = (\nabla\Phi \text{ x } \nabla\Psi)^{-2} \qquad E = W^{-2}(\nabla\Psi)^2 = W^{-2}\Delta_1\Psi$$

$$F = -W^{-2}\nabla\Phi.\nabla\Psi = W^{-2}\Delta_1(\Phi,\Psi) \ , \quad G = W^{-2}(\nabla\Phi)^2 = W^{-2}\Delta_1\Phi \quad (4)$$

1.4 Image of streamlines of aligned MHD flow of a surface S in the Beltrami surface $\overline{S}$

We consider steady plane magneto hydrodynamic flow of a viscous incompressible fluid of infinite electrical conductivity. The governing equations are –

$$\text{div } \mathbf{V} = 0 \quad (5)$$

$$(\mathbf{V}.grad)\mathbf{V} + \rho^{-1} \text{ grad P} = \nu \text{ div(gard } \mathbf{V}) + \mu_e\rho^{-1}(\text{curl } \mathbf{H})\text{x } \mathbf{H} \quad (6)$$

$$\text{div } \mathbf{H} = 0 \quad (7)$$

where **V**=velocity vector , **H** =Magnetic field vector , P=pressure function

ν = kinematic viscosity, μ_e = Magnetic permeability, ρ = fluid density

In this section, we study aligned flows for which the magnetic lines lie in the flow plane and are everywhere parallel to the streamlines. Thus **V** x **H**=0 which implies

$$\boldsymbol{H} = \alpha\boldsymbol{V} \quad and \quad \boldsymbol{V}.grad\,\alpha = \mathbf{0} \quad (8)$$

where α is called abnormality parameter and it is either constant or a function of position and time. Let $\boldsymbol{V} = V(\xi,\eta)$ (9)

In natural, i.e., streamline or curvilinear co-ordinates (ξ, η) with $g_1(\xi,\eta)d\xi$ and $g_2(\xi,\eta)d\eta$ as the components of a vector element of arc length, we have –

$$div\, V = (g_1 g_2)^{-1} \frac{\partial}{\partial \xi} (g_2\, V) \tag{10}$$

$$curl\, \boldsymbol{V} = -(g_1 g_2)^{-1} \frac{\partial}{\partial \eta} (g_2\, V) e_3$$

$$curl\, \boldsymbol{V} \mathrm{x} \boldsymbol{V} = -\boldsymbol{V} (g_1 g_2)^{-1} \frac{\partial}{\partial \eta} (g_2\, V) e_2$$

$$curl\, \Omega = g_2^{-1} \frac{\partial \Omega}{\partial \eta} e_1 - g_1^{-1} \frac{\partial \Omega}{\partial \xi} e_2$$

$$curl\, \boldsymbol{H} \mathrm{x} \boldsymbol{H} = -\alpha V (g_1 g_2)^{-1} \frac{\partial}{\partial \eta} (g_1\, \alpha V) e_2$$

$$\boldsymbol{V}.grad\ \alpha = g_1^{-1} V \frac{\partial \alpha}{\partial \xi}$$

$$grad \left(P\rho^{-1} + \frac{\boldsymbol{V}^2}{2} \right) = g_1^{-1} \frac{\partial}{\partial \xi} \left(P\rho^{-1} + \frac{\boldsymbol{V}^2}{2} \right) e_1 += g_2^{-1} \frac{\partial}{\partial \eta} \left(P\rho^{-1} + \frac{\boldsymbol{V}^2}{2} \right) e_2$$

Where V is the magnitude of velocity e_1 a unit vector in the direction of the velocity, e_2 a unit vector in the direction perpendicular to the velocity but in the plane of flow, e_3 a unit vector in the direction of normal to the plane of flow such that e_1 , e_2 and e_3 form a right handed co-ordinate system. Using equations (10) in (5), (7), (8) we have-

$$\frac{\partial}{\partial \xi} (g_2\, V) = 0 \tag{11}$$

$$\frac{\partial \alpha}{\partial \xi} = 0 \tag{12}$$

$$\Omega = -(g_1 g_2)^{-1} \frac{\partial}{\partial \eta} (g_1\, V) \tag{13}$$

$$V \frac{\partial V}{\partial \xi} + \rho^{-1} \frac{\partial P}{\partial \xi} + g_1 g_2^{-1} \frac{\partial \Omega}{\partial \eta} = 0 \tag{14}$$

$$V^2 g_1^{-1} \frac{\partial g_1}{\partial \eta} - \rho^{-1} \frac{\partial P}{\partial \eta} + g_2 \nu g_1^{-1} \frac{\partial \Omega}{\partial \xi} = \mu_e (\rho g_1)^{-1} \alpha V \frac{\partial}{\partial \eta} (g_1 \alpha V) \quad (15)$$

Let us consider the flow in a surface S. the arc length of any streamline in the surface in curvilinear co-ordinate system (ξ, η) is given by (a).

We assume that the streamlines in the surface S are straight lines but not parallel and envelope a curve C and their orthogonal trajectories, the involutes of C, as a system of curvilinear co-ordinates. The squared element of arc length dS in this orthogonal curvilinear co-ordinate system is given by [9],

$$dS^2 = d\xi^2 + \big(\xi - \sigma(\eta)\big) d\eta^2 \quad (16)$$

where $\sigma = \sigma(\eta)$ denotes the arc length of the curve C and η- the angle subtended by the tangent line with x-axis and ξ- a parameter constant along the involute of this co-ordinate system. Comparing (a) and (16), we have –

$$g_1^2 = E = 1, \quad F = 0, \quad g_2^2 = G = \big(\xi - \sigma(\eta)\big)^2, \quad W^2 = EG - F^2 = \big(\xi - \sigma(\eta)\big)^2 \quad (17)$$

One can easily see that E and G of (17) satisfy Gauss characteristic equation.

$$K = \frac{-1}{\sqrt{EG}} \left[\frac{\partial}{\partial \xi} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial \eta} \right) \right] = 0 \quad (18)$$

Hence the flow in the surface S is planar and the flow is laminar.

Now differentiating (14) with respect to η and (15) with respect to ξ and adding we obtain –

$$\frac{\partial}{\partial \eta} \left(V \frac{\partial V}{\partial \xi} \right) + \nu \frac{\partial}{\partial \eta} \left(\frac{g_1}{g_2} \frac{\partial \Omega}{\partial \eta} \right) + \nu \frac{\partial}{\partial \xi} \left(\frac{g_2}{g_1} \frac{\partial \Omega}{\partial \xi} \right) + \frac{\partial}{\partial \xi} \left(\frac{V^2}{g_1} \frac{\partial g_1}{\partial \eta} \right) = \mu_e \rho^{-1} \frac{\partial}{\partial \xi} \left(\frac{\alpha V}{g_1} \right) \frac{\partial}{\partial \eta} (g_1 \alpha V) \quad (19)$$

Using (11) we have –

$$V = g_2^{-1} A(\eta) \quad (20)$$

where A(η) is an arbitrary function η. As discussed above for an orthogonal system in the Beltrami surface, we must have

$$\nabla \Phi . \nabla \psi = 0 \quad (21)$$

Using (3) and (17) in (21) we have –

$$\phi_2 = -\left(\xi - \sigma(\eta)\right)^2 \phi_1 \psi_1 \psi_2^{-1} \tag{22}$$

By equation (17), (3), (22) in (4) we obtain

$$E = \psi_2^2 \phi_1^{-2} \left[\psi_2^2 + \left(\xi - \sigma(\eta)\right)^2 \psi_1^2\right]^{-1} = g_1^2 \tag{23}$$

$$G = \left(\xi - \sigma(\eta)\right)^2 \left[\psi_2^2 + \left(\xi - \sigma(\eta)\right)^2 \psi_1^2\right]^{-1} = g_2^2$$

$$W = \psi_2^2 \phi_1^{-2} \left[\psi_2^2 + \left(\xi - \sigma(\eta)\right)^2 \psi_1^2\right]^{-2}$$

Substituting g_1 and g_2 from (23) in equations (11) and (13) with $g_1 = \boldsymbol{g_1}, g_2 = \boldsymbol{g_2}$ we have

$$v = A(\eta)^{-1}\left(\xi - \sigma(\eta)\right)^{-1} \left[\psi_2^2 + \left(\xi - \sigma(\eta)\right)^2 \psi_1^2\right]^{1\backslash 2} \tag{24}$$

$$\Omega = A(\eta) \left[\psi_2^2 + \left(\xi - \sigma(\eta)\right)^2 \psi_1^2\right] \left(\xi - \sigma(\eta)\right)^{-2} [\psi_{22}\psi_2^{-1} + A^{-1}(\eta)A'(\eta) - \phi_1^{-1}\phi_{12} + \sigma'(\eta)(\xi - \sigma(\eta))]$$

Using (23), (24) in (19), after simplification to hold it identically we must have $\xi = \sigma(\eta)$ gives us $v\psi_2^2\sigma'^2(\eta)A(\eta)(1 - 8\,\phi_1\,\psi_2) = 0$ (25)

On the curve $\xi = \sigma(\eta)$ we have by (22), $\phi_2\psi_2 = 0$. This gives either $\phi_2 = 0$ or $\psi_2 = 0$ or $\phi_2 = \psi_2 = 0$.

Case (i) : Let $\phi_2 = 0$ and $\psi 2 \neq o$

$\phi_2 = 0$ implies $\phi = f(\xi)$.This with equation (25), we have-

$$v\psi_2^2\sigma'^2(\eta)A(\eta)(1 - 8\,\phi_1\ f'(\xi)) = 0$$

Since $v \neq 0, \psi_2 \neq 0,\ A(\eta) \neq 0$ we have either

(i) $\sigma'(\eta) = 0$ i.e. $\sigma(\eta) =$ Constant that is the arc length is same in all directions from a fixed point. Hence the streamlines in the Beltrami surface are concentric circles, therefore in a complex lamellar flow, the streamlines are concentric circles

or (ii) $\left(1 - 8\,\phi_1\, f'(\xi)\right) = 0.$

i.e. $\left(f'(\xi)\right) = \frac{1}{\sqrt{8}}$ or $\phi = f(\xi) = \frac{\xi}{\sqrt{8}}$.

Case (ii): $\phi 2 \neq 0$ and $\psi_2 = 0$

The equation (25) holds identically, since $\psi(\xi,\eta)$ = constant is a distant function for the family of Beltrami surfaces $\phi(\xi,\eta) =$ constant and $\psi_2 = 0$implies that in a Beltrami surface the streamlines are radial.

Case (iii) $\psi 2 = 0$ and $\phi 2 = 0$

This is same as case (ii) discussed above.

Conclusion: Differential geometry has been of increasing importance to mathematical physics due to Einstein's general relativity postulation that the universe is curved. Differential geometry technique is one of the powerful method in solving nonlinear differential equations. The field of astronomy, especially mapping the positions of the stars and planets on the celestial sphere and describing the relationship between movements of celestial bodies, served as an important source of geometric problems during the next one and a half millennia Study of images on different surfaces has lot of applications in engineering, Medial and in nature .Images of an object on an eye ball, scanning a body, image of an object in mirror of the vehicles etal are some applications of this study. We conclude the above discussion by the following

Theorem : If the streamlines in steady, plane, aligned MHD flow on a surface S are involutes of a curve C then their images in a Beltrami surface $\overline{\boldsymbol{S}}$ are concentric circles or $\phi = f(\xi) = \frac{\xi}{\sqrt{8}}$ or radial .

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