Research Paper Mathematics

Numerical Solution of Boussinesq's Equation in Ground Water Infilteration Phenomenon by Differential Quadrature Method

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ABSTRACT

Boussinesq's equation is the non-linear partial differential equation of groundwater infiltration phenomenon which has been solved with appropriate boundary conditions by using Differential Quadrature Method (DQM). The results of DQM solution were compared with the results obtained from the Explicit Finite Differences Method (EFDM) and Implicit Finite Differences Method. Based on the comparison results, it was concluded that the DQM provides similar results but with relatively faster calculation speed, less nodes and memory usage.

KEYWORDS

Boussinesq's equation, Groundwater, Infiltration, Differential Quadrature Method

INTRODUCTION

The water infiltration plays an important role to control salinity of water; contamination of water and agriculture purpose and it is also useful in chemical engineering, nuclear waste disposable problems. This phenomenon has been discussed by Darcy(1856), A.E.Scheidegier (1960), M.Muskat (1946) and Jacob Bear (1979). This phenomenon has also been discussed in case of infiltration of water from head reservoir to tail reservoir in case of homogeneous porous media as well as heterogeneous porous media by Verma (1967) , Mehta & Verma(1977) , Mehta & Patel(2007) and Mehta & Desai(2010). Such problems are also useful to measure moisture content of water in vertical one dimensional ground water recharge and dispersion of any fluid in porous media. It has been discussed by M.N.Mehta (2006), Mehta & Patel (2007), Mehta & Yadav (2007), Mehta & Joshi (2009), Mehta & Mehar (2010), Parikh A.K et.al (2011) from different viewpoints.

The one-dimensional unsteady flow equation for groundwater flow is a parabolic differential equation, and typically, Finite Differences Methods (FDM) and Finite Elements Method (FEM) are used for the numerical solution. In the solution of differential equations, DQM can be used as an alternative to these conventional methods. In the homogeneous or heterogeneous soil, confined or unconfined aquifer problems are common in Hydrogeology, Civil Engineering, Irrigation and Drainage Engineering. During recent years, the studies and research on the aquifer problems related to the initial and boundary conditions have been successfully carried out; meanwhile, many computational method and techniques have been developed for numerical solution of the governing equations. Lockington (1997) has studied the response of unconfined aquifer to sudden change in boundary head. In his paper, the analytical approximations to the solution of the one-dimensional Boussinesq's equation were obtained using a weighted residual method. Wang and Anderson (1995) have studied several groundwater flow problems, in a finite aquifer with recharge boundary, by using an explicit finite difference method for numerical solution. Onder (1997) has found an analytical solution for one of the problems that Wang and Anderson had, by examining the flow resulting from a sudden rise or decline in the water stage of a flood channel in a composite aquifer. Also, Finite Element Method (FEM) has found a wide range of applications in groundwater investigations. Khebchareon and Saenton (2005) used Crank-Nicolson and Galerkin Finite Element Method for 1D groundwater. Liou and Yeh (1997) investigated a one-dimensional groundwater transport equation with two uncertain parameters, groundwater velocity and longitudinal dispersivity.

STATEMENT OF A PROBLEM

Infiltration is the process by which water on the ground surface enters the soil. The purpose of this paper is to present physically meaningful technique to determine effective height of the water table as a measure of initial storage capacity of a basin. Thus first an equation is derived for mean water table height on the basis of hydraulic theory of ground water by means of Boussinesq's equation; second is to solve that equation by using Differential Quadrature Method.

MATHEMATICAL FORMULATION

For mathematical formulation of the problem, following are the assumptions:

(i) The stratum has height and lies on top of a horizontal impervious bed, which is labeled as z = 0;
(ii) Ignore the transversal variable y; and
(iii) The water mass which infiltrates the soil occupies a region described as

$\Omega = \{(x, z) \in R: z \le h(x, t)\}$.

Clearly, $0 \le h(x, t) \le$, maximum height of free surface and the free boundary function h is also an unknown. For the sake of simplicity and for the practical computation after introducing a suitable assumption, the hypothesis of almost horizontal flow, i.e., we assume that the flow has an almost horizontal speed.

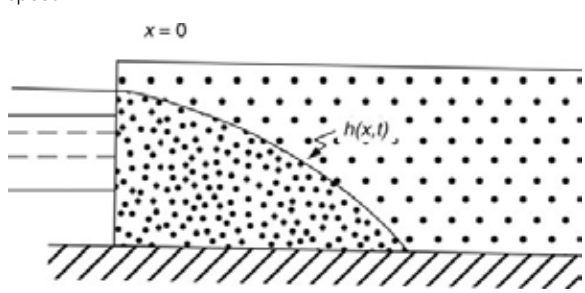


Figure: 1. Ground infiltration

u $\approx$ (u, 0), so that h has small gradients. It follows that in the vertical component

of the momentum equations

$$\rho\left(\frac{du_z}{dt} + u \cdot \nabla u_z\right) = -\frac{\partial p}{\partial z} - \rho g$$

Neglecting the inertial term (the left-hand side), Integration in z gives for this first approximation $p + \rho g z =$ constant.

Now calculate the constant on the free surface z = h(x, t).

If we impose continuity of the pressure across the interface, we have p = 0 (assuming constant atmospheric pressure in the air that fills the pores of the dry region z > h(x, t)).

$\Rightarrow$ p = ρg (h – z). (1)

In other words, the pressure is determined by means of the hydrostatic approximation.

Now using mass conservation law, taking a section S = (x, x + a) × (0, C),

$$\varepsilon \frac{\partial}{\partial t} \int_{x}^{x+a} \int_{0}^{h} dydx = -\int_{\partial s} u \cdot n \, dl \qquad (2)$$

where ε is the porosity of the medium, i.e., the fraction of volume available for

the flow circulation, and u is the velocity, which obeys Darcy's law in the form

that includes gravity effects

$$u = -\frac{k}{\mu} \nabla (p + \rho g z) \qquad (3)$$

On the right-hand lateral surface we have u · n ≈ (u, 0) · (1, 0) = u,

i.e., – (k/μ)px, while on the left-hand side we have –u.

Using the formula for p and differentiating in x, we get

$$\varepsilon \frac{\partial h}{\partial t} = \frac{\rho g k}{\mu} \frac{\partial}{\partial x} \int_{0}^{h} \frac{\partial}{\partial x} h \, dz \qquad (4)$$

We thus obtain Boussinesq's equation.

$$\frac{\partial h}{\partial t} = \beta \frac{\partial^2}{\partial x^2} \left(h^2 \right) \qquad (5)$$

where constant $\beta = \rho g k / \varepsilon \mu$.

SOLUTION OF A PROBLEM:

Choose dimension less variable X = $\frac{x}{1}$ and T = 2 β t, then equation (5) can be written as

$$2 \frac{\partial h}{\partial T} = \frac{\partial^2}{\partial X^2} \left(h^2 \right) \qquad (6)$$

With the boundary conditions

h(0, T) = h_m when x = 0 , any T > 0 (7)

h(L,T) = h_0 when X=L , any T> 0 (8)

equation (6) can be written as

$$\frac{\partial h}{\partial T} = h \frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial x} \right)^2$$

Using Differential Quadrature Method,

$$\frac{dh_i}{dT} = h \sum_{j=1}^{N} b_{ij} h_j + \left(\sum_{j=1}^{N} a_{ij} h_j \right)^2 \quad where \; i = 1,2,3, \ldots N$$

In the present study, Polynomial Differential Quadrature Method (PDQM) is used and the coefficients of weight matrices were calculated by use of Quan and Chang's Approach [1989a, 1989b]. In the distribution, grid points are more frequently near boundaries (non-equally). Chebyshev-Gauss-Lobatto grid points distribution is more appropriate in the time dependent problems as discussed by Change Shu[1999]. Therefore, in the DQM solutions, Chebyshev-Gauss-Lobatto grid-points distribution was used, because the problem is depended on the time.

The initial and boundary conditions for the solution of Equation (6) which is written for one dimensional flow are h(x-,0)=16m, h(0,T)=16 m and h(L,T) =11 m. The same example was solved use of analytical method and numerical method by Onder (1997) and the results were compared.

In the numerical solutions, different numbers of grid points were used. The differences between analytical solution and numerical solutions are calculated, and in the different times, Root Mean Square Error values for DQM, IFDM and EFMD are given in Table 1. For different number of grid points RMS Error can be written as follows:

where is the difference between analytical solution and numerical solution. Results of DQM, IFDM and EFDM solutions are given in Fig. 2 for some number of grid points. In the EFDM, some unstable results can occur depending on the soil hydraulic properties and the size of the spatial and temporal mesh. Thus, some results can't be obtained in EFDM accordingly.

$N_x \times N_t$	DQM			IFDM			EFDM		
	50 sec	100 sec	500 sec	50 sec	100 sec	500 sec	50 sec	100 sec	500 sec
11x11	0.0000	0.000002	0.000181	0.0621	0.113	0.0181	0.0361	0.0196	0.002111
11x51	0.0000099	0.00000964	0.00001888	0.0196	0.0139	0.00615	0.00644	0.0078	0.002133
11x101	0.00000726	0.00001079	0.00002439	0.0151	0.0124	0.00598	0.000697	0.000856	0.00067183
101x1001				0.0133	0.00126	0.000282	0.000347	0.000215	
101x2001				0.00752	0.00211	0.000559	0.000691	0.0000832	

Table 1 RMS Error values

As seen in the table, results of DQM convergence rapidly as increasing Nt values. DQM solution use of few number of grid points, the results are obtain closer to analytical solution. Also, RMS Error values at DQM are smaller than the other numerical methods. Furthermore, the less grid points are used in DQM.

5. CONCLUSION

A DQM approach has been used in the groundwater hydraulics in this study. DQM has found increasing use in recent years in Hydraulic Engineering, because it is an alternative approach to the conservative methods. From the previous applications of DQM, it is seen that the results of DQM are converged rapidly and closer to analytical solutions than other numerical solutions. In present study, one dimensional groundwater problem is solved with appropriate boundary conditions. The DQM results were then compared with the results obtained from IFDM and EFDM. The study has shown that the DQM is quickly converged and closer to analytical solution than the other methods. In comparison, the results of DQM at 11x11 grid points are similar to results at 101x1001 grid points in IFDM and at 11x101 grid points in EFDM. The use of relatively fewer nodes with an acceptable accuracy and advantage of the method, in order to save compile time and memory usage.

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