Research Paper Mathematics

Partial Orderings on CON-S-K-EP Matrices

Dr. G.Ramesh | Associate Professor, Ramanujan Research Centre, Department of Mathematics, Govt. Arts College (Autonomous), Kumbakonam-612 001.

Dr. B.K.N.Muthugobal | 4, Noor Nagar, Kumbakonam- 612 001, India.

ABSTRACT

In this paper, conditions for all those matrices that lie below or above a given con-s-k-EP matrix relative to a particular matrix partial ordering to be con-s-k-EP are determined.

KEYWORDS Con-s-k-EP matrix, Partial ordering.

1. Introduction

Let c_{nxn} be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For Aϵc_{nxn} . Let $\bar{A}$, A^T, A^*, A^S , $\bar{A}^S$, $A^\dagger$, R(A), N(A) and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose , conjugate secondary transpose, Moore-Penrose inverse range space, null space and rank of A respectively. A solution X of the equation AXA=A is called generalized inverse of A and is denoted by A^-. If A ϵc_{nxn} then the unique solution of the equations A XA =A, XAX=X, $[AX]^*$=AX, $(XA)^* = XA$ **[6]** is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$. A matrix A is called Con-s-k-EP$_r$ if $\rho(A) = r$ and N(A)=N(A^TVK) (or) R(A)=R(KVA^T). Throughout this paper let "$\mathcal{k}$" be the fixed product of disjoint transposition in S_n {1,2,....n} and k be the associated permutation matrix .

Let us define the function $\mathcal{k}$ (x)= $\left(x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)}\right)$. A matrix A=(a_{ij}) ϵ c_{nxn} is s-k-symmetric if a_{ij} =$a_{n-k(j)+1,n-k(i)+1}$ for i, j= 1,2,.....n . A matrix A ϵ c_{nxn} is said to be con-s-k-EP if it satisfies the condition $A_x = 0 <=> A^s \mathcal{k}(x) = 0$ or equivalently N(A) =N(A^TVK). In addition to that A is con-s-k-EP <=> KVA is con-EP or AVK is con-EP and A is con-s-k-EP <=> A^T is con-s-k-EP$_r$ moreover A is said to be con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer **[2]**.

Definition 2 (Result 2.1[1])

For $A \in C_{n\times n}$ and K is the permutation matrix associated with the fixed transposition 'k' in S_n the following is equivalent:

(i) A is k - hermitian.
(ii) $A = KA^*K$.
(iii) KA is hermitian.
(iv) AK is hermitian.
(v) A is hermitian relative to K.

Theorem 2.1

If $A \geq_L B$, B is con-s-k-EP and $N(A) \subseteq N(B)$, then A is con-s-k-EP.

Proof

Since $A \geq_L B$, by **Theorem (2.1.6) [5]**,

$KVA \geq_L KVB$, $KVA - KVB \geq 0$, $KV(A-B) \geq 0 \Rightarrow (A-B)$ is con-s-k-hermitian positive semi definite.

By **Definition (2),**

$$(A-B)^T = KV(A-B)VK,$$

since B is con-s-k-EP, by **Theorem (2.11) [2]** $B^T = HKVBVK$.

Now,

$$A^T - B^T = KVAVK - KVBVK$$

$$A^T = KVAVK - KVBVK + B^T$$

$$= KVAVK - KVBVK + HKVBVK$$

$$= KVAVK + (H-I)KVBVK$$

$$= KVAVK + LKVBVK$$

where $L = H - I$

$$A^TVK = KVA + LKVB.$$

Since $N(A) \subseteq N(B)$, it follows that $N(KVA) \subseteq N(LKVB)$, for any $x \in N(A), Ax = 0 \Rightarrow$

$$A^TVKx = KVAx + LKVBx = 0.$$

Therefore, $N(A) \subseteq N(A^TVK)$ and

$\rho(A) = \rho(A^TVK) \Rightarrow N(A) = N(A^TVK)$

$\Rightarrow A$ is con-s-k-EP.

Theorem 2.2

If $A \geq_{rs} B, B$ is con-s-k-EP and $(A-B)$ is con-s-k-hermitian, then A is con-s-k-EP.

Proof

Since $(A-B)$ is con-s-k-hermitian and B is con-s-k-EP, as in the above proof we get, $A^TVK = KVA + LKVB.$

$$A \geq_{rs} B \Rightarrow B = BA^-A$$
$$\Rightarrow N(A) \subseteq N(B) = N(LKVB)$$

Therefore, for any $x \in N(A), Ax = 0 \Rightarrow A^TVKx = 0.$

Hence, $N(A) \subseteq N(A^TVK)$ and $\rho(A) = \rho(A^TVK) \Rightarrow N(A) = N(A^TVK).$ Thus A is con-s-k-EP.

Corollary 2.3

If $A \geq_L B, A \geq_{rs} B$ and B is con-s-k-EP, then A is con-s-k-EP.

Proof

$$A \geq_L B \Rightarrow KVA \geq_L KVB$$
$$\Rightarrow KVA - KVB \geq 0$$
$$\Rightarrow KV(A-B) \geq 0$$
$$\Rightarrow (A-B)$$ is con-s-k-hermitian and then A is con-s-k-EP directly follows from **Theorem (2.2).**

Corollary 2.4

If $A \geq_T B$, B is con-s-k-EP and $(A-B)$ is con-s-k-hermitian, then A is con-s-k-EP.

Proof

This follows from **Theorem (2.2)** and **Result (2.1.2) [5].**

Next the concept of p.s.d., h.p.s.d., a.d and a.p.d **[3, 4]** or **[3]** are generalized respectively to p.con-s-k.s.d., h.p.con-s-k.s.d., a.con-s-k.d and a.p.con-s-k.d as follows.

Definition 2.5

$A \in C_{n\times n}$ is said to be positive con-s-k-semi definite (p.con-s-k.s.d) if $\mathrm{Re}\left(X^T KVAx\right) \geq 0$ for $x \in C_n$.

Definition 2.6

If A is con-s-k-hermitian and p con-.s-k.s.d, then A is h.p.con-s-k.s.d denoted as $A \geq_{s-k} 0$.

Definition 2.7

$A \in C_{n\times n}$ is said to be almost positive con-s-k definite (a.con-s-k.s.d) if for

$$x \in C_n, x^T KVAx = 0 \Rightarrow KVAx = 0 \Rightarrow Ax = 0.$$

Definition 2.8

$A \in C_{n\times n}$ is said to be almost positive con-s-k definite (a.p.con-s-k.s.d) if it is both a.con-s-k.d and p.con-s-k.s.d.

Theorem 2.9

Let A be con-s-k-EP and p.con-s-k.s.d and B be any matrix, so that $(A-B)$ is con-s-k-hermitian, then $A \geq_T B \Rightarrow A \geq_{rs} B \Rightarrow A \geq_L B$.

Proof

By **Result (2.1.3) [5]**, $A \geq_T B \Rightarrow A \geq_{rs} B$. Now we need only to prove the second implication.

$$A \geq_{rs} B \Rightarrow B = BA^\dagger B.$$

$$= BA^\dagger A = AA^\dagger B$$

(by Result (2.1.4) [5])

$$A - B = A - 2B + B$$

$$= A - B - B + B$$

$$= A - BA^\dagger A - AA^\dagger B + BA^\dagger B$$

$$= AA^\dagger A - AA^\dagger B - BA^\dagger A + BA^\dagger B$$

(since $A = AA^\dagger A$)

$$= AA^\dagger (A-B) - BA^\dagger (A-B)$$

$$= (AA^\dagger - BA^\dagger)(A-B)$$

$$= (A-B)A^\dagger (A-B)$$

$$= KV\left(A-B\right)^T VKA^\dagger \left(A-B\right)$$

(since $A-B$ is con-s-k-hermitian)

$$VK\left(A-B\right) = \left(A-B\right)^T VKA^\dagger \left(A-B\right)$$

$$= \left(KV\left(A-B\right)\right)^T A^\dagger VKKV\left(A-B\right)$$

$$= \left(KV\left(A-B\right)\right)^T \left(KVA\right)^\dagger \left(KV\left(A-B\right)\right)$$

Since A is con-s-k-EP and p.con-s-k.s.d, KVA is con-EP and p.s.d and $(KVA)^{\dagger}B$ is p.s.d, $KV(A-B)$ $=\left(KV(A-B)\right)^{T}\left(KVA\right)^{\dagger}\left(KV(A-B)\right)$ is p.s.d. $(A-B)$ is p.con-s-k.d. Also $(A-B)$ is con-s-k-hermitian.

Therefore,

$(A-B)\geq_{s-k} 0 \Rightarrow KV(A-B)\geq 0 \Rightarrow (A-B)\geq 0 \Rightarrow A\geq_{L}$

Thus $A\geq_{T} B \Rightarrow A\geq_{rs} B \Rightarrow A\geq_{L} B.$

Remark 2.10

The conditions $(A-B)$ to be con-s-k-hermitian is essential can be seen by the following example.

Example 2.11

Let $A=\begin{pmatrix}0&0&1\\0&1&0\\0&0&0\end{pmatrix}, B=\begin{pmatrix}0&0&0\\0&1&1\\0&0&0\end{pmatrix}$

for k=(1,2)(3) where $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$

and $V=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$.

Now $KV=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$ and

$A-B=\begin{pmatrix}0&0&1\\0&0&-1\\0&0&0\end{pmatrix}, KV(A-B)=\begin{pmatrix}0&0&-1\\0&0&-1\\0&0&1\end{pmatrix}\not\geq 0$

$KVA-KVB=\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}$

$=\begin{pmatrix}0&0&-1\\0&0&0\\0&0&1\end{pmatrix}\not\geq 0.$

$(A-B)$ is not con-s-k-hermitian.

KVA is p.s.d $\Rightarrow A$ is p.con-s-k.s.d and

$\rho(A-B)=1 \quad =2-1=\rho(A)-\rho(B)$

$\Rightarrow A\geq_{rs} B.$

Theorem 2.12

If $A\geq_{T} B$ with B con-s-k-EP. Then A is a.p.con-s-k.d $\Rightarrow B$ is a.p.con-s-k.d.

Proof

A is a.p.con-s-k.d $\Rightarrow KVA$ is a.p.d.

$\Rightarrow (KVA)^{\dagger}$

is a.p.d.

$\Rightarrow P^{T}\left(KVA\right)^{\dagger}P$ is a.p.d for any P.

Now, $A \geq_T B \Rightarrow A \geq_{rs} B$

(by Result (2.1.3) [5])

$\Rightarrow B = BA^{\dagger}B$

(by Result (2.1.4) [5])

$\Rightarrow B^T = HKVBKV$

(by Theorem (2.11) [2])

$= HKV(BA^{\dagger}B)KV$

$= HKVBKVVKA^{\dagger}BKV$

$= (HKVBKV)(VKA^{\dagger})(BKV)$

$= (HKVBKV)(AKV)^{\dagger}(BKV)$

$= B^T\left(AKV\right)^{\dagger}\left(BKV\right)$

$\Rightarrow VKB^T = VKB^T\left(AKV\right)^{\dagger}\left(BKV\right)$

$\Rightarrow \left(BKV\right)^T = \left(BKV\right)^T\left(AKV\right)^{\dagger}\left(BKV\right)$

$\Rightarrow \left(BKV\right)^T$ is a.p.d.

$\Rightarrow VKB^T$ is a.p.d.

$\Rightarrow B^T$ is a.p.con-s-k.d

$\Rightarrow B$ is a.p.con-s-k.d.

Remark 2.13

The Converse of **Theorem (2.12)** need not true. For example, let $A = \begin{pmatrix} -1 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for k=(1,2)(3), the permutation matrix K be $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ B is con-s-k-EP and a.p.con-s-k.d. $A \geq_T B$. However A is not con-s-k-EP. Hence A is not a.p.con-s-k.d.

Remark 2.14

The conditions on B to be con-s-k-EP in **Theorem (2.12)** cannot be relaxed. For consider the matrices.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $KV = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$,

$KVA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, KVB = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Clearly, $KA \geq_{rs} KB \Rightarrow A \geq_{rs} B$. A is h.p.con-s-k.s.d. Hence A is a.p.con-s-k.d. But B is not con-s-k-EP. Hence B is not a.p.con-s-k.d.

Theorem 2.15

If $A \geq_L B$, A is con-s-k-EP and $N(B) \subseteq N(A)$. Then B is con-s-k-EP.

Proof

Similar to that of **Theorem (2.1)** the following corollary can be deduced from **Theorem (2.1)** and **Theorem (2.15).**

Corollary 2.16

If $A \geq_L B$, and $N(A) \subseteq N(B)$, then A is con-s-k-EP $\Leftrightarrow B$ is con-s-k-EP.

REFERENCES

[1] Hill, R. D., Bates, R.G. and Waters, S.R., "On centro hermitian matrices." SIAM J. Matrix Anal. Appl., 11 (1990), 128-133. | [2] Krishnamoorthy,S., Gunasekaran, K. and Muthugobal, B.K.N., "Con-s-k-EP matrices", Journal of Mathematical Sciences and Engineering Applications , Vol. 5, No.1, 2011, 353 – 364. | [3] Lewis, T.O. and Neuman, T.G., "Pesudo inverse of positive semi definite matrices." SIAM J. Appl. Math., 10 (1968), 701-703. | [4] Meenakshi, A.R., "On the partial ordering of almost definite matrices." Czechoslovak Math. J., 39 (1989), 169-175. | [5] Ramesh, G. and Muthugobal, B.K.N., "Con-s-k-Invariant Partial Orderings on Matrices", IOSRJM,Submitted. | [6] Rao, C.R. and Mitra, S.K., "Generalized Inverse of Matrices and Its Applications", Wiley and Sons, New York, 1971. |