Research Paper Engineering

Polar Angle in Terms of Jacobi Hyperbolic Functions

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ABSTRACT

In mathematics Jacobi elliptic functions are the set of basic elliptic functions that have many applications in Physics, Geometry, Number theory, Mechanics & Engineering. These functions satisfy a system of Differential equations. In particular a solution of Non-Linear Differential equations for the Polar Angle of a point of an ellipse can be expressed in terms of Jacobi elliptic functions.

The aim of the present paper is to briefly present Non-Linear Differential equation for the Polar Angle of a point of a hyperbola and to express the solution of this Differential equation in terms of Jacobi hyperbolic function.

KEYWORDS Polar Angle, Jacobi Elliptic function, Jacobi Hyperbolic function.

1. Introduction

The theory of Jacobi elliptic functions arose in an attempt to integrate certain algebraic expressions, but soon found many applications to geometry, mechanics, physics and engineering. The theory of these important functions is vast. The Jacobi elliptic functions are important examples of doubly periodic meromorphic functions that appear in variety of problems in physics and engineering. Many of their basic properties come as immediate applications of the fundamental theorems on existence, uniqueness and continuous dependence of solutions on initial conditions and thus this system of equations serves as an excellent example in a course in Differential equations. Non-Linear Differential equations with solution expressed in terms of Jacobi elliptic functions occur in many areas of physics such as the study of fields in wave guides, anharmonic oscillations, the period of simple pendulum, the nonlinear poission Boltzmann equation in connection with flows in fluids and plasmas(1-5).the functions have been studied for many years starting back in the 18th century, and a detailed and strong basis of the theory has been developed starting from the Weierstrassian elliptic functions, the theta functions to the Jacobi elliptic functions(6,7). In 2012 Rim Johannessen derived a Non-Linear Differential equation for the polar angle of a point of an ellipse. He also expressed the solution of this differential equation in terms of Jacobi elliptic functions dn(u,k)

In the present paper a Non-linear Differential equation for the polar angle of a point of hyperbola is derived and the solution of this Differential equations is expressed in terms of the Jacobi Hyperbolic function dh(u,k),that at the same time result in new and interesting relationships.

2. Jacobi Hyperbolic functions

With the argument and modulus of hyperbolic functions defined, the functions themselves are just ratios, as in case of trigonometry

shu = sh(u, k) = y
chu = ch(u, k) = x/a
dhu = dh(u, k) = r/a

2.1 Subtraction Theorems

The functions satisfies the two algebraic relations

$ch^2(u, k) - sh^2(u, k) = 1$
$dh^2(u, k) - k^2 sh^2(u, k) = 1$

2.2 The formulas for differentiating hyperbolic functions.

$$\frac{d}{du}chu = shu\ dhu$$

$$\frac{d}{du}shu = chu\ dhu$$

$$\frac{d}{du}dhu = k^2 shu\ chu$$

3 Expression of Polar Angle in terms of Jacobi Hyperbolic function

Equation of hyperbola in Cartesian coordinates is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where a is semi-major axis and b is semi-minor axis. The eccentricity or the modulus k of hyperbola is given by $k = k2 - 1$ where $k' = b/a$ is the complementary modulus. Let us furthermore express a particular point of this hyperbola in terms of a parameter u, so that its Cartesian coordinates are $x(u) = a \cdot ch(u)$ and $y(u) = b \cdot sh(u)$, where $sh(u)$ and $ch(u)$ are Jacobi hyperbolic functions. (The dependence on the modulus is implicit in the following except when explicit dependence is of importance). The polar radius of the point {x(u),y(u)} is given by the relationship

$r^2(u)=x^2(u)+y^2(u)$.

Thus Polar radius of this point can be expressed as $r(u)=a.b.dh(u)$.

It should be emphasized that a value of $b = 1$, should always be applied, and this is assumed throughout this paper. However, for clarity the dependence on b is maintained in several expressions. From these relationships, it follows directly that

$$\frac{dx(u)}{du} = a[sh(u).dh(u)] = \frac{1}{b^2} r(u)\,y(u) \quad (i)$$

And

$$\frac{dy(u)}{du} = b[ch(u).dh(u)] = \frac{1}{a^2} r(u)x(u) \quad (ii)$$

And from the relationship

$$r^2(u) = x^2(u) + y^2(u)$$

$$\frac{dr(u)}{du} = x(u)y(u)\left[\frac{1}{a^2} + \frac{1}{b^2}\right]$$

Now, let us introduce the polar angle $\theta(u)$ corresponding to the same particular point of the hyperbola by the equation $\tan\theta(u) = y(u)/x(u)$ (iii). By differentiation of (iii) and at the same time making use of the equation of the hyperbola as well as (i, ii), one obtains

$$\frac{d\theta(u)}{du} = \frac{1}{r(u)} = \frac{1}{a.b\ dh(u)}$$

Differentiating one more time, one has

$$\frac{d^2\theta(u)}{du} = -\frac{1}{r^2(u)}.\frac{dr(u)}{du}$$

$$= -\frac{1}{r^2(u)}\left[\frac{1}{a^2}+\frac{1}{b^2}\right]x(u).y(u)$$

$$= -\left[\frac{1}{a^2}+\frac{1}{b^2}\right]cos\theta(u).sin\theta(u)$$

$$= -\frac{1}{2}\left[\frac{1}{a^2}+\frac{1}{b^2}\right]sin2\theta(u) \quad (*)$$

With the solution

$$\theta(u) = \frac{1}{a.b}\int_0^u \frac{1}{dh(u)}d(u)$$

It should be noted that since d h (u) is a positive periodic function, the function $\theta(u)$ is an increasing function but not periodic, whereas $\theta'(u)$ is periodic. From elementary trigonometry, it follows that

$$\theta(u) = \arccos\frac{x(u)}{r(u)} \quad (iv)$$

So that

$$\frac{1}{a.b}\int_0^u \frac{1}{dh(u)}d(u) = \arccos\frac{ch(u)}{b.dh(u)} \quad (v)$$

One can equivalently express this integral

$$\frac{1}{a.b}\int_0^u \frac{1}{dh(u)}d(u) = \arcsin\frac{sh(u)}{a.dh(u)} \quad (vi)$$

$$\frac{1}{a.b}\int_0^u \frac{1}{dh(u)}d(u) = \arctan\frac{b.sh(u)}{a.ch(u)} \quad (vii)$$

Equations v, vi & vii are easily proven to be consistent, and going back, one can easily verify that they satisfy (*). In terms of k and k', the important results of this section would be

$$\frac{d\theta(u)}{du} = \frac{1}{a.b\ dh(u)} = \frac{k'}{dh(u)} \quad (viii)$$

$$\frac{d^2\theta(u)}{du} = \frac{k^2}{2}\sin 2\theta(u)$$

$$\theta(u) = k'\int_0^u \frac{1}{dh(u)}d(u)$$

$$\int_0^u \frac{1}{dh(u)}d(u) = \frac{1}{k}\arccos\frac{ch(u)}{dh(u)}$$

$$\int_0^u \frac{1}{dh(u)}d(u) = \frac{1}{k}\arcsin\frac{k'sh(u)}{dh(u)}$$

$$\int_0^u \frac{1}{dh(u)}d(u) = \frac{1}{k}\arctan\frac{k'.sh(u)}{ch(u)}$$

where the Jacobi hyperbolic functions are implicitly dependent on k.

Since the derivative of the amplitude function is given as a $\frac{d\,am(u)}{du} = dh(u)$, it follows from (viii) that the derivative of the polar angle is inversely proportional to the derivative of the amplitude function.

4. Conclusions

The result of above study indicates that Jacobi hyperbolic functions are flexible functions that appear in variety of problem in differential equations and geometry. A new non-Linear Differential equation for the Polar Angle was derived and its solution is expressed in terms of Jacobi hyperbolic functions. Thus Jacobi Hyperbolic functions have many applications in the field of differential equations and Coordinate geometry. Jacobi hyperbolic functions open up a window of solvable non-linear differential equations, all of which relate to physical problem and physical phenomenon.

Acknowledgments

I dedicated my work to my husband for his full support.

REFERENCES

1. Z.Fu, S.Liu, S. Liu, and Q. Zhao, "New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations," Physics Letters A, vol. 290, no. 1-2, pp. 72–76, 2001. | 2. Á.Martín Sánchez, J.Díaz Bejarano, and D.Caceres Marzal, "Solution of the anharmonic quartic potential oscillator problem," Journal of Sound and Vibration, vol. 161, no. 1, pp. 19–31, 1993. | 3. A.Belendez, C.Pascual, D.I. Mendez, T.Belendez, and C. Neipp, "Exact solution for the nonlinear pendulum," Revista Brasileira de Ensino de Fisica, vol. 29, no. 4, pp. 645–648, 2007. | 4. S. H. Behrens and M. Borkovec, "Exact Poisson Boltzmann solution for the interaction of dissimilar charge-regulating surfaces," Physical Review E, vol. 60, no. 6, pp. 7040–7048, 1999. | 5. D. Gurarie and K. W. Chow, "Vortex arrays for sinh-Poisson equation of two-dimensional fluids: equilibria and stability," Physics of Fluids, vol. 16, no. 9, pp. 3296–3305, 2004. | 6. E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Cambridge University Press, 4th edition, 1980. | 7. L.M. Milne-Thomson, Handbook of Mathematical Functions, Dover Publications, New York, NY, USA, 1972, Edited by: M. Abramowitz and I. A. Stegun. |