Research Paper Chemistry

Notes on (Q, L) - Fuzzy Subgroups of A Group

* S. Sahaya Arockia Selvi ** K. Arjunan

* Department of Mathematics, St.Michael College of Engineering & Technology, Kalayarkoil-630551, Tamilnadu, India.

** Department of Mathematics, H.H.The Rajahs College, Pudukkottai – 622001 Tamilnadu, India.

ABSTRACT

In this paper, we study some of the properties of (Q, L)-fuzzy subgroup of a group and prove some results on these. 2000 Ams Subject Classification : 03F55, 08A72, 20N25.

Keywords : (Q, L) - fuzzy subset, (Q, L) - fuzzy subgroup, (Q, L) - fuzzy relation, Product of (Q, L) - fuzzy subsets.

INTRODUCTION:

After the introdution of fuzzy sets by L.A.Zadeh[19], several researchers explored on the generalization of the notion of fuzzy set. Azriel Rosenfeld[4] defined a fuzzy groups. Asok Kumer Ray[3] defined a product of fuzzy subgroups and A.Solairaju and R.Nagarajan[16, 17, 18] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. We introduce the concept of (Q, L)-fuzzy subgroup of a group and established some results.

1.PRELIMINARIES:

1.1 Definition:

Let X be a non-empty set and L = (L, ≤) be a lattice with least element 0 and greatest element 1 and Q be a non-empty set. A (Q, L)-fuzzy subset A of X is a function A : XxQ → L.

1.2 Definition:

Let (G, +) be a group and Q be a non empty set. A (Q, L)-fuzzy subset A of G is said to be a (Q, L) - fuzzy subgroup (QLFSS) of G if the following conditions are satisfied:

(i) A(x+y, q) ³ A(x, q) Ù A(y, q),
(ii) A(-x, q) ³ A(x, q), for all x and y in G and q in Q.

1.3 Definition:

Let A and B be any two (Q, L)-fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by AxB, is defined as AxB ={ ⟨ ((x, y), q), AxB((x, y), q) ⟩ / for all x in G and y in H and q in Q }, where AxB((x, y), q) = A(x, q) □ B(y, q).

1.4 Definition:

Let A be a (Q, L)-fuzzy subset in a set S, the strongest (Q, L) - fuzzy relation on S, that is a (Q, L)-fuzzy relation V with respect to A given by V((x, y), q) = A(x, q) Ù A(y, q), for all x and y in S and q in Q.

2 – PROPERTIES OF (Q, L)-fuzzy subGROUPS:

2.1 Theorem:

If A is a (Q, L)-fuzzy subgroup of a group (G, +), then A(x, q) ≤ A(e, q), for x in G, the identity e in G and q in Q.

Proof:

For x in G, q in Q and e is the identity element of G. Now, A(e, q) = A(x-x, q) ³ A(x, q) Ù A(-x, q) = A(x, q). Therefore, A(e, q) ³ A(x, q), for x in G and q in Q.

2.2 Theorem:

If A is a (Q, L)-fuzzy subgroup of a group (G, +), then A(x-y, q) = A(e, q) gives A(x, q) = A(y, q), for x and y in G, e in G and q in Q.

Proof:

Let x and y in G, the identity e in G and q in Q. Now, A(x, q) = A(x-y+y, q) ³ A(x-y, q)ÙA(y, q) = A(e, q)ÙA(y, q) = A(y, q) = A(x-(x-y), q) ³ A(x-y, q)Ù A(x, q) = A(e, q)ÙA(x, q)= A(x, q). Therefore, A(x, q) = A(y, q), for x and y in G and q in Q.

2.3 Theorem:

A is a (Q, L)-fuzzy subgroup of a group (G, +) if and only if A(x-y, q) ³ A(x, q) Ù A(y, q), for all x and y in G and q in Q.

Proof:

Let A be a (Q, L)-fuzzy sub group of a group (G, +) and x, y in G, q in Q. Then, A(x-y, q)³ A(x, q)Ù A(-y, q)³ A(x, q)ÙA(y, q). Therefore, A(x-y, q)³ A(x, q)ÙA(y, q), for all x and y in G and q in Q. Conversely, if A(x-y, q) ³ A(x, q) Ù A(y, q), replace y by x, then A(x, q) £ A(e, q), for all x in G and q in Q. Now, A(-x, q)= A(e-x, q) ³ A(e, q)ÙA(x, q) = A(x, q). Therefore, A(-x, q) ³ A(x, q), for all x in G and q in Q. It follows that, A(x+y, q) = A(x-(-y), q) ³ A(x, q) Ù A(-y, q) ³ A(x, q)ÙA(y, q). Therefore, A(x+y, q) ³ A(x, q)ÙA(y, q), for all x and y in G and q in Q. Hence A is a (Q, L)-fuzzy subgroup of G.

2.4 Theorem:

Let A be a (Q, L)-fuzzy subset of a group (G, +). If A(e, q) =1 and A(x-y, q)³ A(x, q)ÙA(y, q), then A is a (Q, L)-fuzzy subgroup of G, for all x and y in G and q in Q, where e is the identity element of G.

Proof:

Let x and y in G, e in G and q in Q. Now, A(-x, q) = A(e-x, q)³ A(e, q)ÙA(x, q) = 1ÙA(x, q) = A(x, q). Therefore, A(-x, q) ³ A(x, q), for all x in G and q in Q. Now, A(x+y, q) = A(x- (-y) , q) ³ A(x, q) Ù A(-y, q) ³ A(x, q) Ù A(y, q). Therefore, A(x+y, q) ³ A(x, q) Ù A(y, q), for all x and y in G and q in Q. Hence A is a (Q, L)-fuzzy subgroup of G.

2.5 Theorem:

If A is a (Q, L)-fuzzy subgroup of a group (G, +), then H = { x / xÎG : A(x, q) = 1} is either empty or is a subgroup of G.

Proof:

If no element satisfies this condition, then H is empty. If x and y in H, then A(x-y, q) ³ A(x, q) Ù A(-y, q) ³ A(x, q) Ù A(y, q) = 1Ù 1= 1. Therefore, A(x-y, q) = 1.

We get x-y in H. Therefore, H is a subgroup of G. Hence H is either empty or is a subgroup of G.

2.6 Theorem:

If A is a (Q, L)-fuzzy subgroup of a group (G, +), then H = {

xÎG: A(x, q) = A(e, q) } is a subgroup of G.

Proof:
Let x and y be in H. Now, A(x-y, q) ³ A(x, q) Ù A(-y, q) ³ A(x, q) Ù A(y, q) = A(e, q) Ù A(e, q) = A(e, q). Therefore, A(x-y, q) ³ A(e, q) ----------- (1). And, A(e, q) = A((x-y) - (x-y), q) ³ A(x-y, q) Ù A(- (x-y), q) ³ A(x-y, q) Ù A(x-y, q) = A(x-y, q).

Therefore, A(e, q)³ A(x-y, q) ---------- (2).

From (1) and (2), we get A(e, q) = A(x-y , q).

Therefore, x-y in H. Hence H is a subgroup of G.

2.7 Theorem:
Let A be a (Q, L)-fuzzy subgroup of a group (G, +). If A(x-y, q) = 1, then A(x, q) = A(y, q), for x and y in G and q in Q.

Proof:
Let x and y in G and q in Q. Now, A(x, q) = A(x-y+y , q) ³ A(x-y, q) Ù A (y, q) = 1Ù A(y, q) = A(y, q) = A(-y, q) = A(-x+x-y, q) ³ A(-x, q) Ù A(x-y, q) = A(-x, q) Ù 1= A(-x, q) = A(x, q). Therefore, A(x, q) = A(y, q), for x and y in G, q in Q.

2.8 Theorem:
Let A be a (Q, L)-fuzzy subgroup of a group (G, +). If A(x-y, q) = 0, then either A(x, q)= 0 or A(y, q)= 0, for all x and y in G and q in Q.

Proof:
Let x and y in G and q in Q. By the definition A(x-y, q) ³ A(x, q) Ù A(y, q)

which implies that 0 ³ A(x, q) Ù A(y, q). Therefore, either A(x, q) = 0 or A(y, q) = 0.

2.9 Theorem:
Let (G, +) be a group and Q be a non-empty set. If A is a (Q, L)-fuzzy subgroup of G, then A(x+y, q) = A(x, q) Ù A(y, q) with A(x, q) ¹ A(y, q), for each x and y in G and q in Q.

Proof:
Let x and y belongs to G and q in Q. Assume that A(x, q) > A(y, q). Now, A(y, q) = A(-x + x + y, q) ³ A(-x, q)Ù A(x + y, q) ³ A(x, q) ÙA(x + y, q) ³ A(y, q) ÙA(x + y, q) = A(y, q). And A(y, q) = A(x, q) Ù A(x+y, q) = A(x+y, q). Therefore, A(x+y, q) = A(y, q) = A(x, q)ÙA(y, q), for all x and y in G and q in Q.

2.10 Theorem:
If A and B are two (Q, L)-fuzzy subgroup of a group G, then their intersection AÇB is a (Q, L)-fuzzy subgroup of G.

Proof:
Let x and y belong to G and q in Q, A = { á (x, q), A(x, q) ñ / x in G and q in Q } and B = { á (x, q), B(x, q) ñ / x in G and q in Q }. Let C =AÇB and C = {á (x, q), C(x, q) ñ / x in G and q in Q }. (i) C(x+y, q) = A(x+y, q) Ù B(x+y, q) ³ {A(x, q) Ù A(y, q)} Ù { B(x, q) Ù B(y, q) }³ {A(x, q) Ù B(x, q) }Ù{A(y, q) Ù B(y, q)}= C(x, q) ÙC(y, q). Therefore, C(x+y, q) ³ C(x, q) Ù C(y, q), for all x and y in G and q in Q. (ii) C(-x, q) = A(-x, q)ÙB(-x, q) ³A(x, q)Ù B(x, q) = C(x, q). Therefore, C(-x, q) ³ C(x, q) , for all x in G and q in Q. Hence AÇB is a (Q, L)-fuzzy subgroup of the group G.

2.11Theorem:
The intersection of a family of (Q, L)-fuzzy subgroups of a group G is a (Q, L)-fuzzy subgroup of G.

Proof:
It is trivial.

2.12 Theorem:
Let A be a (Q, L)-fuzzy subgroup of a group G. If A(x, q)< A(y, q), for some x and y in G and q in Q, then A(x+y, q) = A(x, q) = A(y+x, q), for all x and y in G and q in Q.

proof:
Let A be a (Q, L)-fuzzy subgroup of a group G. Also we have A(x, q) < A(y, q), for some x and y in G and q in Q, A(x+y, q) ³ A(x, q) Ù A(y, q) = A(x, q); and A(x, q) = A(x + y - y, q) ³ A(x + y, q) ÙA(-y, q) ³ A(x + y, q) Ù A(y, q) }= A(x+y, q). Therefore, A(x+y, q) = A(x, q), for all x and y in G and q in Q. Hence A(x + y, q) = A(x, q) = A(y + x, q), for all x and y in G and q in Q.

2.13 Theorem:
Let A be a (Q, L)-fuzzy subgroup of a group G. If A(x, q) > A(y, q), for some x and y in G and q in Q, then A(x + y, q) = A(y, q) = A(y + x, q), for all x and y in G and q in Q.

Proof:
It is trivial.

2.14 Theorem:
Let A be a (Q, L)-fuzzy subgroup of a group G such that Im A={a}, where a in L. If A=BÈC,where B and C are (Q, L)-fuzzy subgroups of G, then either BÍC or CÍB.

Proof:
Let A = B È C = { á (x , q), A(x, q) ñ / x in G and q in Q }, B = {á (x, q), B(x, q) ñ / x in G and q in Q } and C = { á (x, q), C(x, q) ñ / x in G and q in Q }. Suppose that neither BÍ C nor C Í B. Assume that B(x, q) > C(x, q) and B(y, q) < C(y, q), for some x and y in G and q in Q. Then, a = A(x, q) = (BÈC)(x, q) = B(x, q) Ú C(x, q) = B(x, q) > C(x, q). Therefore, a > C(x, q). And, a = A(y, q) = (BÈC)(y, q) = B(y, q) Ú C(y, q) = C(y, q) > B(y, q). Therefore, a > B(y, q). So that, C(y, q) > C(x, q) and B(x, q) > B(y, q).

Hence B(x+y, q) = B(y, q) and C(x+y, q) = C(x, q), by Theorem 2.12 and 2.13.

But then, a = A(x+y, q) = (BÈC)(x+y, q) = B(x+y, q) Ú C(x+y, q) }= B(y, q) Ú C(x, q) < a ---------(1).

It is a contradiction by (1). Therefore, either B Í C or C Í B is true.

2.15 Theorem:
If A and B are (Q, L)-fuzzy subgroups of the groups G and H, respectively, then AxB is a (Q, L)-fuzzy subgroup of GxH.

Proof:
Let A and B be (Q, L)-fuzzy subgroups of the groups G and H respectively. Let x1 and x2 be in G, y1 and y2 be in H. Then (x1, y1) and (x2, y2) are in GxH and q in Q. Now, AxB [(x1, y1) + (x2, y2), q] = AxB((x1+x2, y1+y2), q) = A(x1+x2, q) Ù B(y1+y2, q) ³ {A(x1, q)ÙA(x2, q)}Ù{B(y1, q)ÙB(y2, q)}={A(x1, q) ÙB(y1, q)}Ù{A(x2, q) ÙB(y2, q)}= AxB((x1, y1), q)ÙAxB((x2, y2), q). Therefore, AxB[(x1, y1)+(x2, y2), q] ³AxB ((x1, y1), q) Ù AxB((x2, y2), q). And AxB [-(x1, y1), q] = AxB((-x1, -y1), q)= A(-x1, q) Ù B(-y1, q) ³ A(x1, q)ÙB(y1, q) = AxB((x1, y1), q). Therefore, AxB [-(x1, y1), q] ³ AxB ((x1, y1), q). Hence AxB is a (Q, L)-fuzzy subgroup of GxH.

2.16 Theorem:
Let A and B be (Q, L)-fuzzy subsets of the groups G and H, respectively. Suppose that e and eI are the identity element of G and H, respectively. If AxB is a (Q, L)-fuzzy subgroup of GxH, then at least one of the following two statements must hold.

(i) B(eI , q) ³ A(x, q), for all x in G and q in Q,
(ii) A(e, q) ³ B(y, q), for all y in H and q in Q.

Proof:
Let AxB be a (Q, L)-fuzzy subgroup of GxH.

By contra positive, suppose that none of the statements (i) and (ii) holds. Then we can find a in G and b in H such that A(a, q) > B(eI, q) and B(b, q) > A(e, q), q in Q. We have, AxB((a, b), q) = A(a, q) Ù B(b, q) > A(e, q) Ù B(eI , q) = AxB((e, eI), q). Thus AxB is not a (Q, L)-fuzzy subgroup of GxH. Hence either B(eI , q) ³ A(x, q), for all x in G and q in Q or A(e, q) ³ B(y, q), for all y in H and q in Q.

2.17 Theorem:
Let A and B be (Q, L)-fuzzy subsets of the groups G and H, respectively and AxB is a (Q, L)-fuzzy subgroup of GxH. Then the following are true:

(i) if A(x, q) £ B(eI, q), then A is a (Q, L)-fuzzy subgroup of G.
(ii) if B(x, q) £ A(e, q), then B is a (Q, L)-fuzzy subgroup of G.
(iii) either A is a Q-fuzzy subgroup of G or B is a Q-fuzzy subgroup of H.

Proof:
Let AxB be a (Q, L)-fuzzy subgroup of GxH, x and y in G and q in Q. Then (x, eI) and (y, eI) are in GxH. Now, using the property A(x, q) £ B(eI, q), for all x in G and q in Q, we get, A(x-y, q) = A(x-y, q) Ù B(eIeI , q) = AxB(((x-y), (eIeI)), q) = AxB[(x, eI) + (-y, eI), q] ³ AxB((x, eI), q) Ù AxB((-y, eI), q) = {A(x, q) Ù B(eI ,q)} Ù{A(-y, q) Ù B(eI, q)}= A(x, q) ÙA(-y, q) ³ A(x, q) ÙA(y, q). Therefore, A(x-y, q) ³ A(x, q) Ù A(y, q), for all x, y in G and q in Q. Hence A is a (Q, L)-fuzzy subgroup of G. Thus (i) is proved. Now, using the property B(x, q) £ A(e, q), for all x in H and q in Q, we get, B(x-y, q) = B(x-y, q)Ù A(ee, q) =AxB(((ee) ,(x-y)), q)=AxB[(e, x)+(e, -y), q] ³ AxB((e, x), q) Ù AxB((e, -y), q) ={ B(x, q)ÙA(e, q)}Ù{ B(-y, q) Ù A(e, q)}= B(x, q) Ù B(-y, q) ³ B(x, q) ÙB(y, q). Therefore, B(x-y, q) ³ B(x, q) Ù B(y, q), for all x and y in H and q in Q. Hence B is a (Q, L)-fuzzy subgroup of H. Thus (ii) is proved. (iii) is clear.

2.18 Theorem:
Let A be a (Q, L)-fuzzy subset of a group G and V be the strongest (Q, L)-fuzzy relation of G with respect to A. Then A is a (Q, L)-fuzzy subgroup of G if and only if V is a (Q, L)-fuzzy subgroup of GxG.

Proof:
Suppose that A is a (Q, L)-fuzzy subgroup of G. Then for any x = (x1, x2) and y = (y1, y2) are in GxG and q in Q. We have, V(x-y, q) = V [(x1, x2)-(y1, y2), q] =V((x1-y1 , x2-y2), q) = A((x1-y1), q)ÙA((x2-y2), q) ³{A(x1, q) ÙA(-y1, q)}Ù{A(x2, q) Ù A(-y2, q)}={A(x1, q) ÙA(x2, q)}Ù{A(-y1, q)ÙA(-y2, q)}={A(x1, q)Ù A(x2, q) }Ù {A(y1, q) Ù A(y2, q)}= V((x1, x2), q) Ù V((y1, y2), q) = V (x, q)ÙV (y, q). Therefore, V((x-y), q) ³V(x, q)ÙV(y, q), for all x and y in GxG and q in Q. This proves that V is a (Q, L)-fuzzy subgroup of GxG. Conversely, assume that V is a (Q, L)-fuzzy subgroup of GxG, then for any x=(x1, x2) and y=(y1, y2) are in GxG, we have A(x1-y1, q)ÙA(x2-y2, q) = V((x1-y1 , x2-y2), q) = V [(x1, x2) - (y1, y2), q] = V(x-y, q) ³ V(x, q)ÙV (y, q) = V((x1, x2), q) Ù V ((y1, y2), q) = {A(x1, q) Ù A(x2, q)}Ù{A(y1, q) Ù A(y2, q)}. If we put x2 = y2 = e, where e is the identity element of G. We get, A((x1-y1), q) ³ A(x1, q) Ù A(y1, q), for all x1 and y1 in G and q in Q. Hence A is a (Q, L)-fuzzy subgroup of G.

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