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Geometry Of Geodesic Domes

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ABSTRACT

The shortest path or a straight line joining two points on a sphere is known as geodesic. When a number of points marked on a sphere are joined by geodesics, Geodesic dome is formed. The points on the sphere are required to be as much regular as possible so that the types of geodesics are a minimum, and therefore, the shapes of platonic solids and Archimedean solids derived from platonic solids (1), can be used as the geometry of the geodesic domes. Each platonic solid is associated with three spheres, as is described in chapter on Architectural Geometry (1). This paper discusses the sphere touching the vertices of the platonic solids. The relationship between the length of the sides and the radius of the geodesic dome is of great importance to the architects and engineers for construction of geodesic domes. This relationship has been worked out and mathematical expressions for the lengths of geodesics based on icosahedron and truncated icosahedron are derived and presented in this paper.

Keywords : Geodesic, Geodesic dome, platonic solids, polyhedral, tetrahedron, octahedron, dodecahedron, icosahedron and truncated icosahedron

1.0 Introduction

A lot of work has been done by Archimedes (1) on the platonic solids and other polyhedral carved out of the platonic solids by edge cutting. Buckminster Fuller has contributed a lot in the design and construction of geodesic domes. V. K. Dogra (2) has described a step-by step procedure for the construction of geodesic domes with A4 size waste papers. Still the geometry of the geodesic dome is a riddle to many of the engineers and architects. In this paper the geometry of the geodesic domes, based on two types of polyhedron has been worked out using mathematical calculations. The relationship between the radius of the geodesic dome and the sides of the polyhedron are worked out. The sizes of the triangles based on four divisions of each side of the polyhedron and projecting these points on the spherical surface are also worked out.

1.1 Platonic solids

There are five basic platonic solids as shown in figure 1, whose faces are equilateral triangles or squares or regular pentagons. These platonic solids with number of their faces, edges and vertices are as follows:

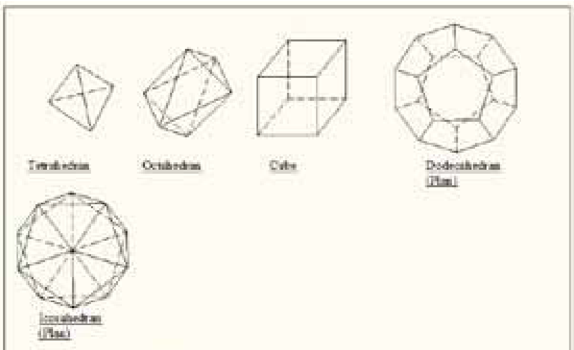

FIG. 1 BASIC PLATONIC SOLIDS (5 NOS.)

GEOMETRICAL PROPERTIES OF BASIC PLATONIC SOLIDS

S. No	Platonic solid	No. of vertices	No. of edges	No. of faces
1	Tetrahedron	4	6	4
2	Cube	8	12	6
3	Octahedron	6	12	8
4	Dodecahedron	20	30	12
5	Icosahedron	12	30	20

The relationship between the number of faces, vertices and edges of platonic solids, as given by Euler is as follows:

$$v - e + f = 2$$

Where v is the number of vertices

e is the number of edges and

f is the number of faces

These platonic solids have their faces as regular congruent polygons and the number of faces meeting at each vertex as same.

More polyhedrons can be formed by cutting the corners of the basic platonic solids. One of these can be truncated icosahedron obtained by cutting the vertices of the icosahedron.

2.0 Icosahedron

Icosahedron is a platonic solid with 20 equilateral triangles. All the 12 vertices of the icosahedron are at the same distance from the centre. Thus a sphere can be drawn which touches all the twelve vertices of the icosahedron. The 30 sides of the icosahedrons can be projected on the sphere thus drawn and the arcs can be further subdivided into two, three, four or more segments. The intermediate points can be joined to the corresponding points on other sides of the equilateral triangle to form smaller triangles. Since all the points of intersection of lines / vertices are on the surface of the sphere, geodesic dome gets formed.

2.1 Relationship between radius of sphere and length of side of equilateral triangle of icosahedron

If we take twenty equilateral triangles and keep on joining them along their sides, a dome gets formed. To make a geodesic dome we can take 30 rods of the same length and keep on joining them with five at every joint, a fully spherical geodesic dome of some diameter gets formed, and it is very simple, and as such there is no need to establish the relationship between the radius and side of the triangle. But the problem is that the engineers and architects are required to start from the diameter / radius of the geodesic dome based on the plan area required to be covered, and then workout the dimensions of the geodesics. Hence the relationship between the radius and the length of the sides of the equilateral triangle becomes important.

Let s be the length of side of equilateral triangle and r be the radius of the sphere passing through all the vertices of the icosahedran under consideration as shown in figure 2, such that PA=PA'=PA''=PA'''=PA''''= AA'=A'A''=A''A'''=A'''A''''= s . From triangle AOD we have

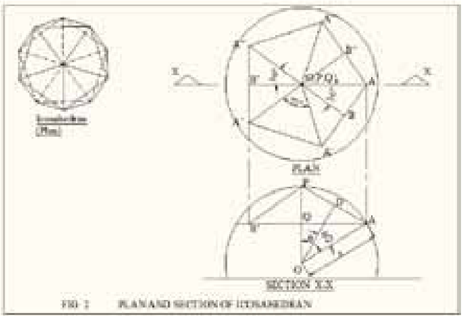

FIG. 2 PLAN AND SECTION OF ICOSAHEDRAN

$$\frac{AD}{AO} = Sin\left(\theta/2\right)$$

Or $\frac{s}{2r} = Sin\left(\theta/2\right)$ $\quad \because AD = \frac{s}{2}$ and AO=r

Or $\theta = 2\sin^{-1}\left(\frac{s}{2r}\right)$ 1

Also from triangle ABQ

$\frac{AB}{QA} = Sin\left(36^o\right)$ Or $QA = \frac{s}{2Sin\left(36^o\right)}$

Now from triangle QAO

$\frac{QA}{OA} = Sin\left(\theta\right)$ Or $\frac{s}{2r\sin\left(36^o\right)} = Sin\left(\theta\right)$

Substituting for from equation -1 we have

$$\frac{s}{2r\sin\left(36^o\right)} = Sin\left(2Sin^{-1}\left(\frac{s}{2r}\right)\right) \quad ....2$$

By setting $\frac{s}{r} = k$ in equation 2, we get

$$\frac{k}{2\sin\left(36^o\right)} = Sin\left(2Sin^{-1}\left(\frac{k}{2}\right)\right) \quad ....3$$

By solving above equation by hit and trial method, we get the value of $k = 1.05146222423827$.

Thus $\frac{s}{r} = k = 1.05146222423827$, and the value of θ from equation no. 1 works out as 63.43494882^o.

2.2 Geodesic Dome of order 2

In this case the length of each side of equilateral triangle of the icosahedran is divided into two parts, dividing the equilateral triangle into four smaller triangles as shown in figure 3. Three out of four triangles, so obtained, are isosceles and one triangle is equilateral, because of the curved surface of the sphere.

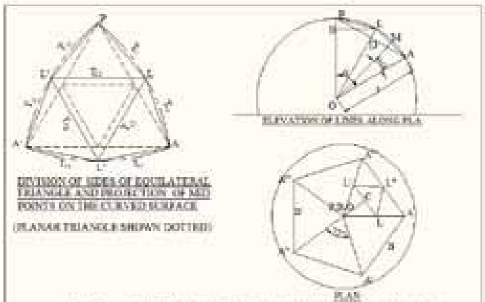

FIG. 3 DIVISION OF EQUILATERAL TRIANGLE INTO 4 TRIANGLES

The additional points on the sides are then projected on the surface of the sphere and the appropriate lengths of the geodesics are worked out as follows:

The division of equilateral triangle into four triangles is shown in figure 3. L is the midpoint of arc PA and M is the midpoint of line LA. From triangle AOM we have

$$\frac{AM}{OA} = Sin\left(\frac{\theta}{4}\right)$$

Or $AM = rSin\left(\frac{\theta}{4}\right)$

Or $T_{11} = 2AM = 2rSin\left(\frac{\theta}{4}\right)$ 4

From triangle BLO

$\frac{BL}{OL} = Sin\left(\frac{\theta}{2}\right)$ Or $BL = rSin\left(\frac{\theta}{2}\right)$

From triangle BLC

$\frac{CL}{BL} = Sin\left(36^o\right)$ Or $CL = rSin\left(\frac{\theta}{2}\right) * Sin\left(36^o\right)$

$$\left\{\because BL = rSin\left(\frac{\theta}{2}\right)\right\}$$

$$LL' = 2 * CL = 2rSin\left(\frac{\theta}{2}\right) * Sin\left(36^o\right)$$

Or $T_{12} = 2rSin\left(\frac{\theta}{2}\right) * Sin\left(36^o\right)$ 5

Once the radius of the geodesic dome is known $s, \theta, T_{11} \,\&\, T_{12}$ can be calculated and the geodesic dome can be easily fabricated. In this geodesic dome 20 triangles are equilateral and the rest 60 triangles are isosceles. There are only two lengths involved in this geodesic dome.

2.3 Geodesic Dome of higher order

In this case the length of each side of equilateral triangle of the icosahedran is divided into three parts, four parts, five parts, …….., dividing each equilateral triangle into 9, 16, 25, …….., smaller triangles. Figure 4 shows four divisions of the sides and the formation of sixteen triangles. One triangle is equilateral, nine triangles are isosceles and six triangles are with all the sides of different lengths.

Additional points on the sides are then projected on the surface of the sphere and the appropriate lengths of the geodesics can be worked out.

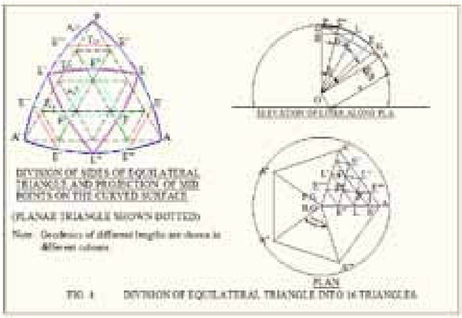

FIG. 4 DIVISION OF EQUILATERAL TRIANGLE INTO 16 TRIANGLES

Geodesics T_{21}, T_{22}, T_{23}, T_{24} and T_{25} for four divisions of the sides of the equilateral triangle as shown in figure 4 can be calculated using following expressions:

$$T_{21} = 2rSin\left(\frac{\theta_1}{4}\right) \qquad ....6$$

where $\left\{\theta_1 = Sin^{-1}\left(\frac{s}{2r}\right)\right\}$

$$T_{22} = 2rSin\left(\frac{Sin^{-1}\left(Sin\theta_1 * Sin\left(36^o\right)\right)}{2}\right) \qquad ....7$$

$$T_{23} = 2rSin\left(\frac{\theta_1}{2}\right)Sin\left(36^o\right) \qquad ....8$$

$$T_{24} = \sqrt{\left(\frac{rSin\theta_1 Cos\left(36^O\right)}{\sqrt{\left(1-\frac{1}{4}Sin^2\theta_1 Sin^2\left(36^O\right)\right)}} - \sqrt{\left(r^2 Sin^2\left(\frac{\theta_1}{2}\right) * \left(1-Sin^2\left(36^O\right)\right)\right)}\right)^2 + \left(\frac{T23}{2}\right)^2}$$

.............9

$$T_{25} = 2\left(\frac{Sin\theta_1 Sin\left(36^O\right) * \sqrt{3} * \sqrt{\left(\frac{1}{Cos\left(30^O\right)} - 1\right)}}{\sqrt{\left(1-\left(Sin\theta_1 Sin\left(36^O\right)\right)^2\right)}} * r\right) Cos\left(30^o\right)$$

.............10

3.0 Truncated Icosahedran

When the vertices of the icosahedran are truncated in such a fashion that regular pentagons and regular hexagons are formed such that the length of each side of hexagon is equal to the length of each side of pentagons, truncated icosahedran is obtained. In other words, the vertices of icosahedron are truncated such that the cutting line is R`R, R'R'' and R'''R`` on all the faces of the equilateral triangles. Thus twelve vertices of icosahedron get converted into twelve pentagons, and twenty faces of twenty equilateral triangles are left as twenty hexagons.

Truncated icosahedran is shown in figure 5.

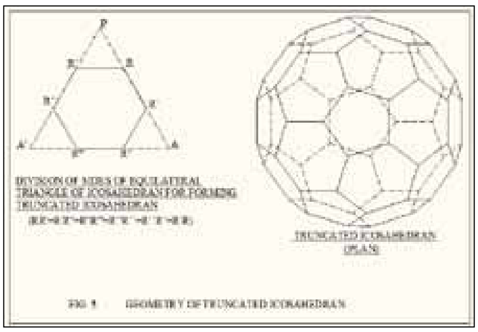

FIG. 5 GEOMETRY OF TRUNCATED ICOSAHEDRAN

3.1 Relationship between radius and length of side of pentagons and hexagons of truncated icosahedron

The relationship between the radius of the geodesic dome and the sides of pentagons / hexagons can be worked out with the help of figure 6 as follows:

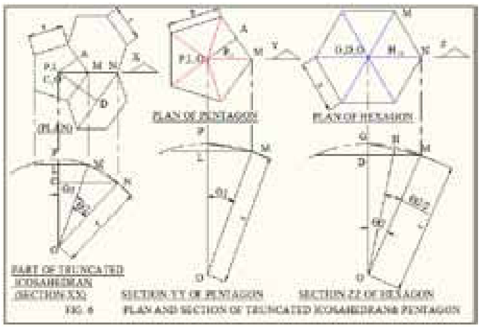

FIG. 6 PLAN AND SECTION OF TRUNCATED ICOSAHEDRAN& PENTAGON

Let us assume that the length of the sides of pentagon and hexagon be s and radius of the sphere passing through all the vertices be r. Consider part of truncated icosahedrons with one pentagon and two adjoining hexagons as above in figure 6.

From triangle LMA

$$\frac{AM}{LM} = Sin\left(36^o\right)$$

Or $LM = \frac{s}{2Sin\left(36^o\right)} \qquad \left\{\because AM = \frac{s}{2}\right\}$

From triangle LMO

$$\frac{LM}{OM} = Sin\theta_1$$

Or $\theta_1 = Sin^{-1}\left(\frac{s}{2rSin\left(36^o\right)}\right)$ 6

$$\left\{\because LM = \frac{s}{2Sin\left(36^o\right)} \& OM = r\right\}$$

From triangle OBN

$$\frac{BN}{ON} = Sin\left(\frac{\theta_2}{2}\right)$$

Or $\theta_2 = 2Sin^{-1}\left(\frac{s}{2r}\right)$ 7

$$\left\{\because BN = \frac{s}{2} \& ON = r\right\}$$

From triangle CND

$$\frac{ND}{CN} = Sin\left(36^o\right)$$

Or $CN = \frac{s}{Sin\left(36^o\right)} \quad \left\{\because ND = s\right\}$

From triangle CNO

$$\frac{CN}{ON} = Sin\left(\theta_1 + \theta_2\right)$$

$$\left\{CN = \frac{s}{Sin\left(36^o\right)}, ON = r\right\}$$

and substituting for $\theta_1 \& \theta_2$ from equations 6 and 7, we get

$$\frac{s}{rSin\left(36^o\right)} = Sin\left(Sin^{-1}\left(\frac{s}{2rSin\left(36^o\right)}\right) + 2Sin^{-1}\left(\frac{s}{2r}\right)\right)$$

Set $\frac{s}{r} = k$, we get

$$\frac{k}{Sin\left(36^o\right)} = Sin\left(Sin^{-1}\left(\frac{k}{2Sin\left(36^o\right)}\right) + 2Sin^{-1}\left(\frac{k}{2}\right)\right)$$

Solving this equation by hit and trial method, we get the value of $k = 0.403548212335198$ (constant).

Or $\frac{s}{r} = 0.403548212335198$

or $s = 0.403548212335198 * r$

or $r = \frac{s}{0.403548212335198}$

or $r = 2.47801865906761 * s$

Thus $\theta_1 = 20^o - 4' - 36.3045888854975''$ &

$\theta_2 = 23^o - 16' - 53.2065847484327''$

3.2 Geodesic Dome of order 1

Next task is to divide these pentagons and hexagons into triangles. Each pentagon can be divided into at least 5 equal triangles and each hexagon can be divided into at least 6 equal triangles as shown in figure 6.

3.2.1 Geometry of Pentagon

Each pentagon is now divided into five triangles by joining the vertices with the centre. If the centre is now raised to touch the surface of the sphere, another vertex is generated. The length of other two sides of the isosceles triangle thus formed can be worked out as follows:

$$LM = \frac{s}{2Sin\left(36^o\right)}$$

From triangle OEM we have

$$\frac{EM}{OM} = Sin\left(\frac{\theta_1}{2}\right)$$

Or $EM = rSin\left(\frac{\theta_1}{2}\right)$

But PM = 2ME

Or $PM = 2rSin\left(\frac{\theta_1}{2}\right)$

Or $P_{11} = 2rSin\left(\frac{\theta_1}{2}\right)$ 8

3.2.2 Geometry of Hexagon

Each hexagon is divided into six triangles by joining the vertices with the centre at D. If the centre is now raised to touch the surface of the sphere at point G, another vertex is generated as shown in figure 6. The length of other two sides of the six isosceles triangles thus formed can be worked out as follows:

From triangle DNO

$$\frac{DN}{ON} = Sin\varphi$$

Or $\varphi = Sin^{-1}\left(\frac{s}{r}\right) \quad \left\{\begin{matrix}\because DN = s \\ \& ON = r\end{matrix}\right\}$

Draw OH as bisector of angle ϕ meeting line GN at H, such thatFrom triangle HNO

$$\frac{HN}{ON} = Sin\left(\frac{\varphi}{2}\right)$$

Or $HN = rSin\left(\frac{\varphi}{2}\right) \quad \{\because ON = r\}$

$GN = 2HN$

Or $GN = 2rSin\left(\frac{\varphi}{2}\right)$

Or $H_{11} = 2rSin\left(\frac{Sin^{-1}\left(\frac{s}{r}\right)}{2}\right)$ 9

3.3 Geodesic Dome of higher order

Each triangle of pentagon can be further divided into four triangles and in this way sixty triangles within twelve pentagons get divided into 60x4=240 smaller triangles. Similarly each triangle of hexagon can be divided into 4 triangles and thus 120 triangles of the twenty hexagons get divided into 120x4=480 smaller triangles. In this type of geodesic dome larger diameter can be achieved with small size members.

Plans of spherical geodesic domes of order 1 and 2 are shown in figure 7.

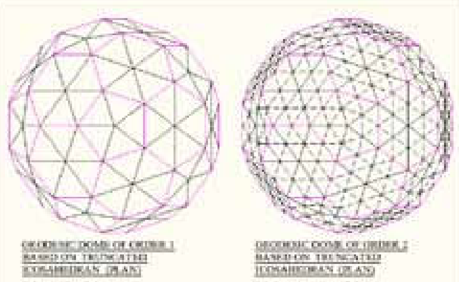

FIG. 7 GEODESIC DOMES OF ORDER 1 AND 2

The lengths of the geodesics for the geodesic dome of order 2 can be calculated using mathematical expressions based on the geometry.

4.0 Conclusions

Mathematical expressions presented in this paper can be directly used to calculate the length of the members of geodesic dome accurately. The riddle of relationship between the radius and the lengths of the sides of geodesic domes, based on the geometry of icosahedron and truncated icosahedron, has been solved, and it can be of tremendous help to the civil engineers and architects in planning the geodesic domes.

REFERENCES

1. Book Extract "Architectural Geometry" Architecture + D E S I G N February 2009, 110 – 124. | 2. Dogra V. K., "Step-By-Step Method of fabrication of Geodesic Dome from A4 Size Waste Papers", Indian Journal of Applied Research, Vol. 1, Issue 3, Year 2011, pp. 20-21.