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Comparing Simple Hypergraphs

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ABSTRACT

This article presents ideas of comparisons of hypergraphs on the basis of number of hyperedges and comparability of hyperedges by set inclusion. Also discussed are ways to refine / coarsen certain classes of Sperner hypergraphs.

Keywords : Hypergraph, hyperedge, Sperner, simple hypergraph.

1. Introduction.

Established terminologies, notations and theorems in set theory and propositional logic [1] are assumed. Let V be a finite nonempty set. Its cardinality (or, size) is denoted by $|V|$. 2^V denotes the power set of V, or the set of all subsets (including the empty set φ) of V. 2^{V*} denotes the set of all nonempty subsets of V; that is, $2^{V*} = 2^V - \{\varphi\}$.

A hypergraph [2] on V is a couple H = (V, E) where V is a nonempty finite set and E is a family of nonempty subsets of V such that $\cup_{X \square E} X = V$. The set V is called the vertex set of H and each member of E is called a hyperedge of H. If no hyperedge in H equals all of V then we call H non-trivial. If the members of E are all distinct (that is, no two members coincide as subsets of V; or, $E \subseteq 2^{V*}$) then H is called simple. If no member of E is a subset (proper or otherwise) of another, then H is called a Sperner hypergraph. In some instances ([2] and [3]) Sperner hypergraphs are taken to be simple and vice versa but there is a distinction [4] between the two: Sperner hypergraphs are necessarily simple but not conversely [6].

All the hypergraphs in this article are assumed non-trivial and simple unless there is some unambiguous indication to the contrary. Motivation for the ideas presented in this article comes from those of fineness and coarseness of topologies [5].

2. Comparisons of hypergraphs

Let $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ be two hypergraphs on the same vertex set V. We say H_2 is *denser* than H_1 (written $H_2 \supseteq H_1$) if$E_2 \supseteq E_1$. In this case we also say H_1 is *rarer* than H_2 (written $H_1 \subseteq H_2$).

We say H_2 is a *refinement* of H_1 (or, E_2 is a refinement of E_1) if: given $A_1 \in E_1$ and $x \in A_1$, there is $A_2 \in E_2$ such that $x \ A_2 \in A_1$. In this case we also say H_1 is a *coarsening* of H_2; or that H_2 is *finer* than H_1, which we write $H_2 > H_1$; or that H_1 is *coarser* than H_2, which we write $H_1 < H_2$. We say two hypergraphs (V, E_1) and (W, E_2) are *equal* if and only if: V = W and $E_1 = E_2$.

2.1: Proposition. (i) Every hypergraph refines itself

(ii) $H_d = (V, 2^{V*})$, the discrete hypergraph on V, refines every hypergraph on V.

(iii) Every hypergraph denser (respectively, rarer) than a given hypergraph H is a refinement (resp., coarsening) of H.

(iv) Two hypergraphs on the same set V are equal if and only if each is denser than the other.

The proof of 2.1 takes only elementary set theory.

2.2: Example. Two hypergraphs on the same set V need not be equal even if each refines the other. Consider $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ where V = {1, 2, 3, 4, 5}, E_1 consists of all the singleton subsets of V and E_2 consists of all the nonempty proper subsets of V. It is a straightforward check that H_1 and H_2 refine each other. Yet $H_1 \neq H_2$, because $E_1 \neq E_2$.

2.3: Proposition. *Let $H_1 = (V, E_1)$, $H_2 = (V, E_2)$ and $H_3 = (V, E_3)$. Then:*

(i) If $H_1 \in H_2$ and $H_2 \in H_3$, then $H_1 \in H_3$

(ii) If $H_3 > H_2$ and $H_2 > H_1$, then $H_3 > H_1$.

Proof. (i) is straightforward.

(ii) Given $A_1 \in E_1$ and $x \in A_1$, there exist $A_2 \in E_2$ and $A_3 \in E_3$ such that $x \in A_2 \subseteq A_1$ and $x \in A_3 \subseteq A_2$. It follows that $H_3 > H_1$

2.4: Proposition. *Let $H_1 = (V, E_1)$ be Sperner. Let $A, B \in E_1$ such that $A \neq B$ and $A \ B \neq V$. Let $E_2 = E_1 \cup \{A \ B\} - \{A, B\}$, and take $H_2 = (V, E_2)$. Then H_2 is coarser than H_1.*

Proof. The fact that $A \cup B \notin E_1$ shows H_2 is simple. Next, suppose $Y \in E_2$ and $x \in Y$ are given. If $Y = A \cup B$, then $x \in A \subset A \cup B$ or $x \in B \subset A \cup B$, with A, B E_1. If $Y \neq A \cup B$, then $Y \in E_1 \cap E_2$, so that $x \in Y \subseteq Y$.

2.5: Proposition. *Given a Sperner hypergraph $H_1 = (V, E_1)$. Suppose there is $A \in E$ such that $A = A_1 \cup A_2$, where A_1 and A_2 are nonempty and disjoint. Let $E_2 = E_1 \cup \{A_1, A_2\} - \{A\}$. Then $H_2 = (V, E_2)$ is finer than H_1.*

Proof. Clearly $A_1 \notin E_1$ and $A_2 \notin E_1$, and so H_2 is simple. Given $X \in E_1$ and $x \in X$. If X = A, then $x \in A_1 \subset A$ or $x \in A_2 \subset A$; else $X \in E_1 \cap E_2$ so that $x \in X \subseteq X$.

3. Summing up

From certain Sperner hypergraphs, coarser hypergraphs can be constructed (2.4); and from certain other Sperner hypergraphs, finer hypergraphs can be constructed (2.5). However, hypergraphs so constructed may not necessarily be Sperner. The authors are studying hypergraph properties that could be preserved as hypergraphs get refined and / or rarified.

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