Research Paper Mathematics

The Global Connected Domination in Fuzzy Graphs

* Dr. S. Vimala ** J. S. Sathya

* Assistant Professor, Department of Mathematics, Mother Teresa Women's University Kodaikanal

** Research Scholar, Department of Mathematics, Mother Teresa Women's University Kodaikanal

Keywords :

I. INTRODUCTION

Most of our traditional tools for formal modelling, reasoning and computing are crisp, deterministic and precise in character. Precision assumes that parameters of a model represent exactly either our perception of the phenomenon modelled or the features of the real system that has been modelled. Now, as the complexity of a system increases our ability to make precise and yet significant statements about its behaviour diminishes until a threshold is reached beyond which precision and significance becomes almost mutually exclusive characteristics. Moreover in constructing a model, we always attempt to maximize its usefulness. This aim is closely connected with the relationship among three key characteristics of every system model: complexity, credibility and uncertainty. Uncertainty has a pivotal role in any efforts to maximize the usefulness of system models. All traditional logic habitually assumes that precise symbols are being employed.

The capability of fuzzy sets to express gradual transition from membership to non membership and vice versa has a broad utility. It provides us not only with a meaningful and powerful representation of measurement of uncertainties, but also with a meaningful representation of vague concepts expressed in natural language. Because every crisp set is fuzzy but not conversely, the mathematical embedding of conventional set theory into fuzzy sets is as natural as the idea of embedding the real numbers into the complex plane. Thus, the idea of fuzziness is one of enrichment, not of replacement. A Mathematical framework to describe the phenomena of uncertainty in real life situation is first suggested by L.A.Zadeh[15] in 1965. The study of dominating sets in graphs was begun by Ore and Berge, the domination number, total domination number are introduced by Haynes and Hedetniemi[5].

Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical subjects like logic, topology, algebra, analysis etc. consequently fuzzy set theory has emerged as potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

The fuzzy definition of fuzzy graphs was proposed by Kaufmann [8], from the fuzzy relations introduced by Zadeh [15]. Although Rosenfeld[11] introduced another elaborated definition, including fuzzy vertex and fuzzy edges. Several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness etc. The concept of domination in fuzzy graphs was investigated by A.Somasundram, S.Somasundram [13]. A. Somasundram presented the concepts of independent domination, total domination, connected domination and domination in Cartesian product and composition of fuzzy graphs([13][14]). In this paper we are presenting lower and upper bounds on $\gamma_{fgc}(G)$ for a general fuzzy graph in terms of its order (n), and we characterize the graphs attaining these bounds.

II. PRELIMINARIES

In this section, basic definitions relating to fuzzy graph are given

Fuzzy set:[9] Let E be the universal set. A fuzzy set A in E is represented by $A = \{(x, \mu_A(x)): \mu_A(x) > 0, x \in E\}$,where the function $\mu_A: E \to [0,1]$ is the membership degree of element x in the fuzzy set A.

Fuzzy Relation:[9] Let S and T be two sets and let μ and ϑ be fuzzy subsets of S and T respectively. Then a fuzzy relation ρ from the fuzzy subset μ into the fuzzy subset ϑ is a fuzzy subset ρ of $S \times T$ such that $\rho(x, y) \leq \mu(x) \wedge \vartheta(y), \forall x \in S \text{ and } y \in T$. That is , for ρ to be a fuzzy relation , we require that the degree of membership of a pair of elements never exceed the degree of memberships of either of the elements themselves.

Fuzzy graph:[9] A fuzzy graph $G(V, \mu, \rho)$ is a pair of function $\mu: V \to [0,1]$ and $\rho: V \times V \to [0,1]$, where for all $u, v \in V$, we have $\rho(u, v) \leq \mu(u) \wedge \mu(v)$.

Fuzzy subgraph: [9] The fuzzy graph $H(V_1, \tau, \sigma)$ is called a fuzzy subgraph of $G(V, \mu, \rho)$ if $V_1 \subseteq V$, $\tau(u) \leq \mu(u)$ for all $u \in V$ and $\sigma(u, v) \leq \rho(u, v)$ for all $u, v \in V$.

Spanning Fuzzy subgraph: [9] The fuzzy subgraph $H(V_1, \tau, \sigma)$ is said to be a spanning fuzzy subgraph of $G(V, \mu, \rho)$ if $\tau(u) = \mu(u)$ for all $u \in V_1$ and $\sigma(u, v) \leq \rho(u, v)$ for all $u, v \in V$. Let $G(V, \mu, \rho)$ be a fuzzy graph and τ be any fuzzy subset of μ, i.e., $\tau(u) \leq \mu(u)$ for all u.

Induced Fuzzy subgraph: [9] The fuzzy graph $H(\vartheta, \tau)$ is called a partial fuzzy subgraph of $G(V, \mu, \rho)$ if $\vartheta \subseteq \mu$ and $\tau \subseteq \rho$. The partial fuzzy subgraph of $G(V, \mu, \rho)$ induced by ϑ is the maximal partial fuzzy subgraph of $G(V, \mu, \rho)$ that has fuzzy vertex set ϑ. This is the partial fuzzy graph (ϑ, τ), where $\tau(x, y) = \vartheta(x) \wedge \vartheta(y) \wedge \rho(x, y) \forall x, y \in V$.

Effective edge:[9] In a fuzzy graph $G(V, \mu, \rho)$, an edge $xy \in X$ is said to be an effective edge $\rho(xy) = \mu(x) \wedge \mu(y)$. Any two vertices of V is said to be effectively adjacent, they have the effective edge between them.

Connected fuzzy graph:[12] Two nodes that are joined by a path are said to be connected. The relation connected is a reflexive, symmetric and transitive, the equivalence classes of nodes under this relation are the connected components of the given fuzzy graph.

Complete fuzzy graph: [9] A complete fuzzy graph is a fuzzy graph $G(V,\mu,\rho)$ such that $\rho(u,v)=\mu(u)\wedge\mu(v)$ for all u and v. We use the notation $C_m(\mu,\rho)$ for a complete fuzzy graph where $|V|=m$.

Complete fuzzy bi-graph: [9] $G(V,\mu,\rho)$ is called a fuzzy bi-graph $\Leftrightarrow$ there exists partial fuzzy sub graphs $(\mu_i,\rho_i), i=1,2\ of\ (\mu,\rho)$ such that $G(V,\mu,\rho)$ is the join $(\mu_1,\rho_1)+(\mu_2,\rho_2)$ where $V_1\cap V_2=\emptyset$ and $X_1\cap X_2=\emptyset$. Fuzzy bi-graph is said to be complete if $\rho(uv)>0$ for all $uv\in X'$. We use the notation $C_{m,n}(\mu,\rho)$ for a complete fuzzy bi-graph such that $|V_1|=m$ and $|V_2|=n$.

Isomorphism in fuzzy graphs:[2] An isomorphism $h: G\to G'$ is a map $h: V\to V'$ which is bijective that satisfies $\mu(x)=\mu'\big(h(x)\big)\ \forall x\in V$

$\rho(x,y)=\rho'\big(h(x),h(y)\big)\forall x,y\in V$ we denote it as $G\cong G'$. That is an isomorphism preserves the weights of the edges and
the weights of the nodes.

Domination in fuzzy graph:[17] Let $G(V,\mu,\rho)$ be a fuzzy graph. A subset D of V is said to be dominating set of G if for every $v\in V-D$ there exists $u\in D$ such that $\rho(u,v)=\mu(u)\wedge\mu(v)$. A dominating set D of a fuzzy graph G is called the minimal dominating set of G if every node $v\in D, D-\{v\}$ is not a dominating set. The minimum scalar cardinality of D is called the domination number and is denoted by $\gamma(G)$.

Global Domination in fuzzy graph:[17] Let $G(V,\mu,\rho)$ be a fuzzy graph. A subset D of V is said to be a global dominating set of G if D is a dominating set of both G and its complement $\bar{G}$.

III. MAIN RESULTS

Definition 1:

For a connected fuzzy graph $G(V,\mu,\rho)$ we call $S\subseteq V(G)$ a global connected domination set of G if S is both global domination and connected dominating set of a fuzzy graph G. The minimum cardinality of a global connected dominating set of G is the fuzzy global connected domination number $\gamma_{fgc}(G)$. A global connected dominating set S is called a minimal dominating set if no proper subset of S is a global connected dominating set. The smallest number of nodes in any global connected dominating set of T is called its global connected domination number and it is denoted by $\gamma_{fgc}(G)$. Note that any global connected dominating set of G has to be connected in G (but not necessarily in $\overline{G}$)

Example:

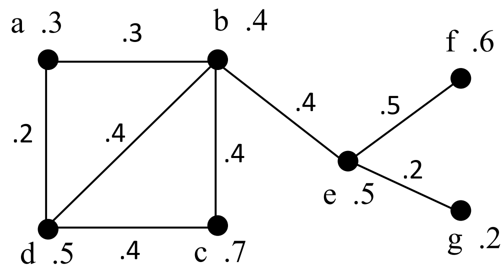

Figure 1.1

In the above example {b,e} is a global dominating set which is also connected in G. Therefore {b,e} is the global connected dominating set of G.

2.Bounds:

Notations:

Fix $n \geq 2$ an integer where n denotes the number of vertices in G. We define a family $\mathcal{F}$ of graphs of order n as follows:

Fix A and B be two disjoint subsets of vertices such that $|A \cup B| = n - 2$. $(A = B = \emptyset \; when \; n = 2)$ and let $a, b \notin A \cup B$. Let $V = A \cup B \cup \{a, b\}$. Denote by $F(A, B) \in \mathcal{F}$ is the fuzzy graph with vertex set V which satisfying the following properties:

1. The vertex a is effectively adjacent to b.
2. The vertex a is effectively adjacent to each vertex in A whenever $A \neq \emptyset$
3. The vertex b is effectively adjacent to each vertex in B whenever $B \neq \emptyset$
4. None of $V(G) - \{a, b\}$ is adjacent to both a and b. Let $F = \bigcup_{A \cap B = \emptyset \; and \; |A \cup B| = n-2} F(A, B)$

Theorem 2:

Let G be a graph of order $n \geq 2$, then

1. $2 \leq \gamma_{fgc}(G) \leq n$
2. $\gamma_{fgc}(G) = n \Leftrightarrow G \cong C_n(\mu, \rho)$
3. $\gamma_{fgc}(G) = 2 \Leftrightarrow G \in \mathcal{F}$

Proof:

For item (1) Note that $1 \leq \gamma_{fgc}(G) \leq n$, it sufficient to show that $\gamma_{fgc}(G) \neq 1$. To the contrary, we may assume that there exist a FGCD set $S = \{v\}$. Hence v is effectively adjacent to all vertices in $V(G) - \{v\}$ and so v is isolated in $\bar{G}$. This contradicts that {v} is a GCD set of G.

For item (2) note that the graph $\overline{C_n(\mu, \rho)}$ consists of n isolated vertices, so the proof of sufficiency is trivial. To show necessity, we take a fuzzy spanning tree of $G(V,\mu,\rho)$ say T. Let v be a leaf of T, and let $S = V(G) - \{v\}$. Note that v is joined to some vertex in S, and the sub graph in G induced by S is connected. Hence S is a connected dominating set of fuzzy graph G, with size $n-1$. As $\gamma_{fgc}(G) = n$, S is not a FGCD set of G. Therefore v is isolated in $\bar{G}$ implying that v is joined to all vertices of S in G. Now we take another spanning tree $T^{'}$ consisting of all edges incident with v. Note that every vertex $u \in V(G) - \{v\}$ is a leaf of $T^{'}$ and $S^{'} = V(G) - \{u\}$ is a fuzzy connected dominating set of G with size $n-1$. By the similar argument, one can show that u is joined to all vertices in $S^{'}$. Hence $G \cong C_n(\mu,\rho)$. For (3) we prove sufficiency first. Let $G = F(A,B) \in \mathcal{F}$. By the construction, we know that $\{a,b\}$ is a global connected dominating set of G. Hence $\gamma_{fgc}(G) \leq 2$ by (1) so, $\gamma_{fgc}(G) = 2$. Now we prove necessity, let G be an arbitrary fuzzy graph with $\gamma_{fgc}(G) = 2$. Suppose that $S = \{a,b\}$ is a FGCD-set of G. Then $ab \in E(G)$. Since S is a FGCD set of G there exists no vertex joined to both a and b. Hence, for each vertex $u \in V(G) - S$, u is joined to either a or b but not both. Denoted by A or B the set of vertices which are joined to a and b, respectively. Clearly $G \in F(A,B)$ and so $G \in \mathcal{F}$.

Note 3:

For all positive integers p and q, $\gamma_{fgc}\left(C_{p,q}(\mu,\rho)\right) = 2$

1. Characterizing fuzzy graphs with $\boldsymbol{\gamma_{fgc}(G) = k (3 \leq k \leq n-1)}$:

In this section, we first characterize the graphs with $\gamma_{fgc} = n-1$.

Theorem 4:

For any fuzzy graph $G(V,\mu,\rho)$ of order $n \geq 3$, $\gamma_{fgc} = n-1 \Leftrightarrow G \cong C_n(\mu,\rho) - e$ where e is a strong edge of $C_n(\mu,\rho)$

Proof:

Sufficiency: If $G \cong C_n(\mu,\rho) - e$, where $e = uv \in X\left(C_n(\mu,\rho)\right)$, then $\bar{G}$ is a graph consisting of an edge uv and $n-2$ isolated vertices. Hence, every global connected dominating set of G must contain all vertices of $V(G) - \{u,v\}$ and at least one of u and v. Hence $\gamma_{fgc} \geq n-1$ the inequality $\gamma_{fgc}(G) \leq n-1$ follows from the fact that $V(G) - \{u\}$ is a FGCD-set of G. Hence $\gamma_{fgc} = n-1$.

Necessity: We may assume that $n \geq 4$, as the assertion holds by theorem 1(3) when $n = 3$. Suppose that S is a FGCD- set of G, v must be effectively adjacent to some vertex in S but not all. Therefore, there exist vertices $x, y \in S$ such that xy is an effective edge and $\rho(yv) = 0$. As S is a FGCD-set of G, there exists a spanning tree T of the induced subgraph $< S >$. Let $w \in S - \{x\}$ be a leaf of T, Since S is a FGCD-set of G with smallest size $n - 1$, $S - \{w\}$ is not a FGCD set of G. Note that $S - \{w\}$ is a connected dominating set of G. Hence, $S - \{w\}$ cannot be a FGD- set of G, which implies that w is not effectively adjacent to any vertex of $S - \{w\}$ in $\bar{G}$. It turns out that w is effectively adjacent to all vertices of $S - \{w\}$ in G. Let T' be the subgraph induced by all edges incident with w in $< S >$. It is clear that T' is a spanning tree of $< S >$ with $n - 2$ leaves. Take any leaf $z \neq x$ on T'. By a similar argument to the above, we can derive that z is joined to all vertices of $S - \{z\}$. Thus $< S > \cong C_{n-1}(\mu, \rho)$. To complete our proof we only need to show that v is effectively adjacent to all vertices of $S - \{y\}$. To the contrary, suppose that there is one vertex $u \in S - \{y\}$, such that $\rho(u, v) < \mu(u) \wedge \mu(v)$. Then $S' = V(G) - \{u, y\}$ is a FGCD-set of G with size $n - 2$, which contradicts the assumption $\gamma_{fgc} = n - 1$.

Lemma 5:

Let G be a fuzzy graph and let S be a fuzzy connected dominating set of G. Then S contains all cut vertices if exists of G.

Proof:

The proof is trivial

Theorem 6:

Let T be a fuzzy tree of order $n \geq 3$. If $T \ncong K_{1,k}$ where $k \geq 2$ is an integer, then $\gamma_{fgc}(T) = n - \beta(T)$ where $\beta(T)$ is the number of leaves of T.

Proof:

Note that T has at least two leaves. Let U be the set of all leaves in T. Since $T \ncong K_{1,k}$, $T - U$ is nontrivial and connected. Note that the fuzzy sub graph induced by $V(T) - U$ is a tree, so it is connected fuzzy graph. As every leaf of T is joined to some vertex in $V(T) - U$ but not all, $V(T) - U$ is a global fuzzy connected domination set of T. Thus $\gamma_{fgc}(T) \leq n - \beta(T)$. Note that every vertex in $V(T) - U$ is a cut vertex in T. By previous lemma, $\gamma_{fgc}(T) \geq n - \beta(T)$. Hence the proof.

Corollary 7:

Let T be a fuzzy tree of order $n \geq 4$, then $\gamma_{fgc}(T) = n - 2 \Leftrightarrow T \ncong P_n$ (fuzzy path with n vertices)

Proof:

We only prove necessity, as sufficiency follows from Theorem 4. Suppose that T is a fuzzy tree of order $n \geq 4$ with $\gamma_{fgc}(T) = n - 2$. By theorem 4, we know that there are exactly 2 leaves in the tree T. It is straight forward to show that any tree with exactly 2 leaves must be isomorphic to a path on n vertices.

Conclusion:

In this work we found some bounds to the new paprameter global connected domination in fuzzy graphs. Characterisation of fuzzy graphs with $\gamma_{fgc}(G) = k(3 \leq k \leq n - 1)$ is also analysed. Global connected domination in fuzzy graphs can be used in World Wide Web to reduce the traffic during communication and simplify the connectivity management**.**

REFERENCES

1. Akram .M, Bipolar fuzzy graphs, Information Sciences 181 (2011) 5548-5564. | 2. Akram M. and Dudek W.A., Regular bipolar fuzzy graphs, Neural Computing & Applications 21 (2012) 197-205. | 3. Akram M. and Karunambigai M.G., Metric in bipolar fuzzy graphs, World Applied Sciences Journal 14 (2011)1920-1927. | 4. Dejan Delic and Changping Wang, The global connected domination in graphs,(print) | 5. Haynes T.W., Hedetniemi S.T., Slater P.J., Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998. | 6. Haynes T.W., Hedetniemi S.T., Slater P.J. (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998. | 7. John N. Mordeson, Premchand S. Nair, Fuzzy graphs and Fuzzy Hypergraphs, Physica-Verlag, Heidelberg, 2001. | 8. Kaufmann.A., (1975), Introduction to the theory of Fuzzy Subsets, Academic Press, Newyork. | 9. Kulli V.R., Janakiram B., Global nonsplit domination in graphs, In: Proceedings of the National Academy of Sciences 11-12 (2005), India. | 10. Nagoor Gani A. and Chandrasekaran V. T. A first look on Fuzzy Graph Theory, Allied Publishers Pvt Ltd ,New Delhi, 2010 | 11. Rosenfield,A., Fuzzy graphs In: Zadeh, L.A., Fu, K.S., Shimura, M.(Eds), Fuzzy sets and their applications(Academic Press, New York) | 12. Sampathkumar E., Global domination number of a graph, Journal Math. Phy. Sci. 23 (1989) 377-385. | 13. Somasundaram .A, Somasundaram S., 1998, Domination in Fuzzy Graphs – I, Pattern Recognition Letters, 19, pp-787-791. | 14. Somasundaram.A, (2004), Domination in Fuzzy Graphs – II, Journal of Fuzzy Mathematics, 20. | 15. Zadeh,L.A.(1971), Similarity Relations and Fuzzy Ordering, Information sciences, 3(2),pp.177-200. |