Research Paper Engineering

Fundamental of Algebra

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ABSTRACT

In this paper we discuss basic properties of algebra for fitting we divided this paper in two sections, section 1 deals with the definitions and section 2 related two properties of algebra.

Keywords:

1. Introduction:

Definition: Let $u \in R^N$ be an N-component column vector.

We say that u is **lexicographically positive** and write $u \succ 0$ iff the first non zero component of u is positive. Next, we say u is **lexicographically negative**, $u \prec 0$ iff –u is lexicographically positive .Further, we say u is **lexicographically nonnegative** ($u \succeq 0$) or nonpositive ($u \preceq 0$) iff ($u \succ 0$ or u = 0),or ($u \prec 0$ or u = 0) respectively .**Notations:** For, u, v $\in R^N$, we denote

(1) $u \succ v \Leftrightarrow u - v \succ 0$

(2) $u \prec v \Leftrightarrow u - v \prec 0$

(3) $u \succeq v \Leftrightarrow u - v \succeq 0$

(4) $u \preceq v \Leftrightarrow u - v \preceq 0$

We shall not assume the commutativity of the field F. F may be skew. Let V be vector space over F. Suppose a binary relation $\preceq$ be given on V and a binary relation $\leq$ be given of F.

Define $u \succeq v$ iff $v \preceq u$. From (3), it follows that $u \succeq 0$ iff $-u \preceq 0$. Further, define $\lambda \geq \mu$ iff $\mu \leq \lambda$ iff $\lambda - \mu \geq 0$. Then it also follows that $\lambda \geq 0$ iff $-\lambda \leq 0$ also $\lambda \leq 0$ iff $-\lambda \geq 0$.

We assume that following statements are true for u, v $\in$ V, $\lambda, \mu \in$ F.

(i) $u \succeq 0 \vee u \preceq 0$

(ii) $u \succeq 0 \wedge u \preceq 0 \Rightarrow u = 0$

(iii) $u \succeq 0 \wedge v \succeq 0 \Rightarrow u + v \succeq 0$

(iv) $\lambda \geq 0 \wedge u \succeq 0 \Rightarrow \lambda u \succeq 0$ and

(v) $\lambda \geq 0 \vee \lambda \leq 0$

(vi) $\lambda \geq 0 \wedge \lambda \leq 0 \Rightarrow \lambda = 0$

(vii) $\lambda \geq 0 \wedge \mu \geq 0 \Rightarrow \lambda + \mu \geq 0$

(viii) $\lambda \geq 0 \wedge \mu \geq 0 \Rightarrow \lambda\mu \geq 0$

Definition: A cone C in a vector space is a set such that u $\in$ C, $\lambda \in$ F, $\lambda > 0$ implies $\lambda u \in$ C .**Observation:** The set $\{u \in V/u \succeq 0\}$ is a cone in V and the set $\{\lambda \in F/\lambda \geq 0\}$ is a cone in F.

Proposition 2.1 The relation $\preceq$ is reflexive, transitive, antisymmetric and total ($u \preceq v$ or $u \succeq v$) .

Proof: We have u - u = 0 . Hence $u \preceq u$. Next , if $u \preceq v$ and $u \succeq v$ then

$u - v \preceq 0$ and $u – v \succeq 0$. By (ii) u - v = 0 . i.e. u = v . For $u \preceq v$, $v \preceq w$, we have $v - u \succeq 0$, $w – v \succeq 0$. By (iii) , (v - u) + (w - v) $\succeq 0$ i.e., $w – u \succeq 0$ i.e., $u \preceq w$. By (i), either $u – v \succeq 0$ or $u-v \preceq 0$.

Note: Similarly, $\leq$ is also reflexive, transitive, antisymmetric & total on F. Observe also that all subsequent propositions hold true if we replace V, $\preceq$, $\succeq$, $\prec$ and $\succ$ by F, $\leq$, $\geq$, < and > respectively.

Remark: A relation which is reflexive, transitive, antisymmetric and total is called linear order. Thus V is a linearly ordered vector space over the linearly ordered field F.

Proposition2.2 (a) If $\lambda \leq 0$ and $u \succeq 0$ then $\lambda u \preceq$ (b) If $\lambda \geq 0$ and $u \preceq 0$ then $\lambda u \preceq$ (c) If $\lambda \leq 0$ and $u \preceq 0$ then $\lambda u \succeq 0$

Proof (aWe know $\lambda \leq 0$ iff $-\lambda \geq 0$. By (iv), $-\lambda u \succeq 0$ .Therefore $\lambda u \preceq 0$

(b) As, $u \preceq 0$ iff $-u \succeq 0$ we get $\lambda(-u) = \lambda u \succeq 0$ by (iv) i.e., $\lambda u \preceq 0$.

(c) We have $\lambda \leq 0$ iff $-\lambda \geq 0$ and u $\preceq$ 0 iff -u $\succ$ 0

This give by (iv), $(-\lambda)$ (-u) $\succ$ 0 <=> λ (-u) $\prec$ 0<=> λ u $\succ$ 0.

Note: We know that 1+ (-1) = 0. Multiply by (-1) on both side to get

(-1).1 + (-1). (-1) = 0

=> $-1 + (-1)^2 = 0$[∵ 1is multiplicative identity)

=> $(-1)^2 = 1$ [∵ 1 is additive identity of -1]

We define two new relations:

1. v $\succ$ u iff u $\prec$ v iff (u $\preceq$ v and u $\neq$ v)
2. $\mu > \lambda$ iff $\lambda < \mu$ iff ($\lambda \leq \mu$ and $\lambda \neq \mu$)

Proposition 2.3: 1 > 0

Proof: Suppose 1 < 0. This means 1 $\leq$ 0 and 1 $\neq$ 0.Thus -1 $\geq$ 0 . By (viii),

-(-1) = (-1) (-1) $\geq$ 0.Note that -(-1) is the additive inverse of (-1) which is 1 Therefore 1 $\geq$ 0. But 1 $\neq$ 0.Hence 1 > 0, which is contradiction.

Proposition 2.4: (a) If $\lambda > 0$ then $\lambda^{-1} > 0$ (b) If $\lambda < 0$ then $\lambda^{-1} < 0$

Proof: Suppose $\lambda^{-1} < 0$. Then $-\lambda^{-1} > 0$. This gives $-\lambda^{-1} = -1 > 0$ which is not true .Similarly we can prove (b) .

Remark: For F = R & V = R^N**,** we can see that the standard ordering of real numbers $\leq$ and the lexicographic ordering of R^N denoted by $\preceq$ satisfy all the above required properties (i) to (viii). That is R^N is a linearly ordered vector space over the linear ordered field R .

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