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The Revision on Fundamentals of Ring II

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ABSTRACT

The study of Ring and Distributive lattice is equivalent to the study of the corresponding algebra, because the Ring has a property of Group and similarly Modular lattice is a special case of Distributive lattice and one can be obtained from other by putting further algebraic approach.So in this paper we discuses the important properties of Ring.

Keywords: waste, disposal of e-waste, e-waste crisis, WEEE, Management

1. Introduction:The great Mathematician Brian C. Hall [18] provedmany important properties in group and similarly in [15] Ralph Freese and Ralph Mckenzie have one important property of Modular lattice. So it is possible to give the concise characterizations of algebra in terms of Ring and Distributive lattice. The study of Ring and Distributive lattice is equivalent to the study of the corresponding algebra, because the Ring has a property of Group and similarly Modular lattice is a special case of Distributive lattice and one can be obtained from other by putting further algebraic approach.First we give some essentials knowledge of Ring theory.

1.1 Ring: If *R* is any non –empty set, and +, ∗ are two binary operations define on this set, then an algebraic structure (*R,* +,∗) is called a ring if it satisfies following properties:

i (*R,* +) is a group.
ii (*R,*∗) is a semi group.

iii An operation∗is a distributive over the operation +(addition).

1.2 Commutative Ring: A Ring (*R*, +,∗)

is called Commutative Ring if it

satisfies Commutative property with respect to ∗ operation.i.e.

a ∗*b=b*∗*a* for all values in *R*.

1. 3 Sub -Ring: A non empty subset of Ring, which satisfy all theproperties of Ring is called a sub- Ring.

1.4Integral domain: A Ring (*R*, +,∗) iscalled an integral

domain if it satisfy following Properties;

i. It is commutative ring i.e. *a* ∗*b=b*∗*a* for all values in *R*.

ii.It is ring with unity i.e.there exist an element 1 in R such that *a*∗1=1∗*a=a* generally it is denoted by*(I,+,* ∗*).*

1. 5 Field : A Ring *(R,+,* ∗*)* is called a Field if it satisfy following Properties

i. It is commutative ring i.e. *a* ∗*b=b*∗*a* for all values in *R*.

ii. It is ring with unity i.e. there exist an element 1 in *R* such that *a*∗*1=1*∗*a=a*

iii. It has inverse for second operation i.e. with respect to operation∗.i.e. if *a* in *R* then there exist *b* in *R* such that *a*∗*b=b*

∗ *a=1.*

Then *a* is an inverse of *b* and *b* is inverse of *a.*

1.6Centre of a group: The centre of a group G is the set of

all $g \in G$ such that g.h=h.g for all $h \in G$

7 Centre of Field: If *F* is a field ,then the set of all $f \in F$

such that *f.h=h.f* for all values of *h* in *F*.

2. Theorems:
2. 1 Theorem : If *R* be a commutative ring and let n be any positive integer then R[n]={$x \in R : nx = 0\}$ is a sub ring

of *R.*

Proof : Consider R[n] = {$x \in R : nx = 0\}$.It is given that R is a commutative ring. And we have to show that R[n] is a sub ring.

Closure property: Let $x_1, x_2 \in R[n]$, a

$x_1, x_2 \in R[n]$ therefore $nx_1 = 0, nx_2 = 0$

$nx_1 + nx_2 = 0 \Rightarrow n(x_1 + .x_2) = 0 \Rightarrow x_1 + .x_2 \in R[n]$

Existence of an Identity: As 0.n=0 $\Rightarrow 0 \in R[n]$ which is an identity element.

Existence of an Inverse: Let $x_1, x_2 \in R[n]$

as $0 \in R$ therefore by closure property $x_1 + .x_2 = 0 \Rightarrow x_1$ is an

inverse of x_2 and x_2 is an inverse of x_1.

2.2 Theorem: Let *I* be an Ideal of commutative Ring R and suppose that *R/I* is Cyclic of finite order q. If $R = \langle I, t \rangle$. then

$R' = \langle I, t \rangle$.

Proof: As R is a Commutative ring.

$R = \langle I,t \rangle . \Phi : a \to (a,t) = a^{-1}a^{t}$ will be endomorphism. And, *I* , *t* are normalizer of $\langle I,t \rangle$. So *R* is also normalizer of $\langle I,t \rangle$ as *R* is Commutative ring and I is an Ideal therefore *R/I* will be also an abelian .which implies $\langle I,t \rangle$ =R'.

2.3 Theorem: The centre of a Field is an Integral .

Proof: Let C*(F)* is a centre of integral domain . i.e C (*F*) = {*f: f.h=h.f , f+h=h+f* for all $h \in F$ } First we prove (C(*F*),+) is a group.

Closure property: Let $f_1, f_2 \in C(F)$ i.e.

$f_1 + h = h + f_1, f_2 + h = h + f_2$

where f_1, f_2 are arbitrary elements in C(F).

Consider $(f_1 + f_2) + h = f_1 + (f_2 + h)$ (Associative in *F)*

⇨ $f_1 + (h+f_2)$

⇨ $(f_1 + h)+f_2$

⇨ $(h+f_1) + f_2$

⇨ $h+(f_1 + f_2)$

⇨ $f_1 + f_2 \in F$

closure property is satisfied.

Existence of an identity element: C(F) is a subset of F. as 0+h=h+0 where $0 \in F$ Therefore $0 \in C(F)$

Existence of an inverse : As $0 \in C(F)$

therefore by closure property *a+b*=0 => *a* is an inverse of *b* and *b* is an inverse of *a*.by

the above properties it is clear that (C(F),+) is a group.

Closure property: Let $f_1, f_2 \in C(F)$ **i.e.** $f_1 h = hf_1, f_2 h = hf_2$

Where f_1, f_2 are arbitrary elements in C(F) Consider $(f_1 f_2)h = f_1(f_2 h)$ (associative in *F*)

⇨ $f_1(hf_2)$

⇨ $(f_1 h)f_2$

⇨ $(hf_1)f_2$

⇨ $h(f_1 f_2)$

⇨ $f_1 f_2$

⇨ $f_1 f_2$

⇨ $f_1 f_2 \in F$

closure property is satisfied.

Commutative Property: As C (F) is a subset of F, and as Commutative Property satisfied in F. Therefore this satisfy also in C(F).

Existence of unit element: $1 \in F$ and as 1.h=h.1=> $1 \in C(F)$

As all above properties are satisfied therefore we can say that C(F) is an Integral domain.

2.4 Theorem: If *A* is a Distributive lattice and let *F* be an n-frame in con*A*,then $|1_F, 1_F| \le 0_F$

Proof : As we know that [15] If V is a variety of algebras with distributive Congruence then these lattices have an additional operation known as the commutator, this is denoted $[\alpha, \beta]$ If $A \in V[\alpha,\beta]\, and, \alpha, \beta, and, \beta_i \in conA$ then i. $[\alpha, \beta] \le \alpha \wedge \beta$

ii. $[\alpha, \beta] = [\beta, \alpha]$

iii. $[\alpha \vee \beta_i] = \vee[\alpha, \beta_i]$

Let $1_F = a_1 \vee .... \vee a_n$ by using above $[1_F, 1_F] \le 0_F \vee \vee[a_i, a_i]$

but

as $a_i \le a_j \vee c_j$ .We have

$[a_i, a_i] \le [a_i, a_j \vee c_j] = [a_i, a_j] \vee [a_i, c_j] \le (a_i \wedge a_j) \vee (a_i \wedge c_j) = a_i \wedge (a_j \vee c_j) = 0_F$

hence $|1_F, 1_F| \le 0_F$

2.5 Theorem: The automorphism of a lattices form a Distributive lattices.

Proof: Let f: *D* $\to D$ be an automorphism .(where *D* is a Lattice).Such that *f(a)=a* for all values of *a* in *D.*

It is evident, f is well define, one-one and onto. Let *f,g* and *h* are three automorphism on Distributive latticeNow, we define

$f \wedge g = f \cap g$ i.e. Intersection of *f* and *g.*

$f \vee g$ *=f.* i.e. *fg= {xy: x*$\in f, y \in g$*}*As, any Lattice satisfy Distributive inequality.i.e. We have, $f \vee (g \cap h) \subseteq (f \vee g) \cap (f \vee h)$

{ We Define $\subseteq = \le$ *}* …(2.5.1)

Now , we have to prove $(f \vee g) \cap (f \vee h) \subseteq f \vee (g \cap h)$

Let, $a \in (f \vee g) \cap (f \vee h)$

$\Rightarrow$ $a \in f \vee g$ and $a \in f \vee h$

$\Rightarrow$ *a* $\in$ *fg* and *a* $\in$ *fh*

$\Rightarrow$ *a =bc* $\in$ *fg* and *a=bc* $\in$ *fh*

$\Rightarrow$ *b* $\in$ *f,c* $\in$ *g* and *b* $\in$ *f, c* $\in$ *h*

$\Rightarrow$ b $\in$ f ,c $\in$ g $\cap h$

$\Rightarrow$ *bc* $\in$ f $\vee (g \cap h)$

$\Rightarrow$ *a=bc* $\in$ f $\vee (g \cap h)$

$\Rightarrow$ (f$\vee g) \cap (f \vee h) \subseteq L \vee (g \cap h)$ …(2.5.2)

From Equation (6.2.5.1) and (6.2.5.2)

(f$\vee g) \cap (f \vee h) = f \vee (g \cap h)$

Therefore it is clear that automorphism on a lattice form a Distributive lattice.

2.6 Theorem: If *L* and *L'* be two lattices ,and Ψ is a homomorphism of *L* into *L'*and *M* is congruence $\approx$,then kernel of Ψ is also a kernel of *M*.

Proof: It is given that L and L' be two lattices. and Ψ :L$\to L'$ be homomorphism.Then kerΨ =set of all those elements whose image is an identity element.i.e. ker Ψ ={*x*: Ψ (*x*)= an identity element }

$$\Psi^{-1}(identityelement) = \{x / \Psi(x) = identityelement\}$$

$$\Rightarrow \{x / \Psi(x) = \Psi(identityelement) = \{x / x \approx identityelement\}$$

$$\Rightarrow [identityelement] = \Theta^{-1}(identityelement) = M$$

2.7 Theorem: An Integral domain which is not relatively complemented has a homomorphic image isomorphic to the Chain of three elements.

Proof : Let *I* is an Integral domain with the property A. And *R* is not relatively complemented, and let *I* has a homo-morphic image isomorphic to the chain of three elements. As *I* has an HI property [1], in this case the chain of three elements also has the property A. This is contradiction and let an Integral domain is not relatively complemented , then there three elements *a,b,c* exist in R such that *b<c<a* and c has no relatively complemented in the interval [*b a*] .Let consider the dual ideal $D = \{d : a \le d \cup c\}$ and the dual ideal E=[*c*)$\cup D$ those elements of *E* which are included in c are in the form of c$\cap d.$ b is not element of E, because if not *b*= *c*$\cap d.$ for all

And so $b = a \cap b = c \cap (a \cap d)$ would hold. It is clear that $a \cap d \in D$ but we have if $b \in E, then$ $a \cap d. \in D$, hence $a \le c \cup (a \cap d)$ thus in effect of *c<a* we get *a=*$c \cup (a \cap d)$ therefore it is clear that $a \cap d$ is a relative complement of *c*.Again we get contradiction,so by Stone ‘s theorem we consider prime ideal and can prove this as in [1], and finally we conclude the three elements of chain will be $0, \alpha, 1$ and 0 is a least and 1 is Greatest element of chain.

2.7.1Corollary: Any finite Field which is not relatively complemented has a homomorphic image isomorphic to the Chain of three elements**.**

Proof : By the definition of Field it is trivial.

2.8 Theorem: If in a Ring [8] there is a one-one correspondence between congruence relations and Ideals then this ring is a relatively complemented ring.

Proof : Let *R* is a ring and there is a one-one correspondence between congruence relations and ideals of this.And we have to prove this ring is relatively complemented. As we know that an ideal of of the ring is the kernel of precisely one homomorphism.

Let *R’* be a homomorphic image of *R* ,and *I’* an ideal in *R’*. if *I’* is the kernel of more than one homomorphism, then its complete inverse image has the same Property .i.e. this has HI property the chain of three elements does not have the above property, because the ideal (0] is a Congruence class under the identical congruence relation. And in which $\alpha \equiv 1, \alpha \ne 0,$

hence it is clear that it will be relatively complemented.

2.8.1 Corollary: If in an Integral domain there is a one-one correspondence between congruence relations and ideals then this Integral domain is a relatively complemented.

2.9 Theorem: Let A be an HI property [9] of Ring [4]. If the semi group (with respect to Operation) of three elements and have the Property A, then this Ring has multiplicative inverse.

Proof: Let R is a ring with HI property and the Semi group (with respect to **.** operation) of three elements and have the property A, but let it has no multiplicative inverse.Consider the semi group which contains three elements *a,*b,c such that b<c<*a .* As it has no multiplicative inverse. that mean c has no multiplicative inverse.Let us consider the Ideal *D* ={*d;a*$\le d + c\}$ and the dual Ideal *E=[c)* $+ D. \cup D.$those elements of *E* which are included in c are in the form of $c \cap d.b$ is not element of *E*, because if not b= c$.d.$ for all $d \in D$ .And so $b = a.b = c(a.d)$ would hold. therefore it is clear that *a* is an identity element of this semi group. But as this is a semi group .therefore by closure property *b.c=a* that means c is an inverse of *b* and *b* is an inverse of *a*. This is contradiction. Therefore it is clear that our assumption is wrong. It means let A be an HI property [1] of Ring. If the semi group(with respect to . operation) of three elements and have the property A then this Ring has multiplicative inverse.

2.10 Theorem: If *X* be a subset in Distributive lattice *L* with 0. Then a map $\Theta : X^{[2]} \to L$ is a Banaschewski measure then

$y\Theta x = (y \wedge z)\Theta(y \wedge x)$ for all $x \le y \le z$

Proof: Let Θ is a Banaschewski measure on X.and also let $x \le y \le z$ in X .And $v := (y \wedge z)\Theta(y \wedge x)$ .obviously $x \wedge v = 0$ Furthermore, as $x \le y$ and by the definition of Distributive lattice $x \wedge v = y \wedge (x \vee z)\Theta(x \vee x) = y \wedge z = y$ .And as $y\Theta x \le v$ and *L* is Distributive lattice therefore $v = y\Theta x$ therefore $y\Theta x = (y \wedge z)\Theta(y \wedge x)$

2.11 Theorem: Let L is a is Distributive lattice with zero,let $e, b \in L$ such that $e \oplus b = 1$ .And let [20] $X \subseteq L \downarrow b.$ if there exist an L- valued Banaschewski Function on $e \oplus X := \{e \oplus x : x \in X\}$,then there exist a $(L \downarrow b)$-valued[20] Banaschewski function on *X.*

Proof : Let Θ is a Banaschewski measure on $e \oplus X$ .Then we have $y\Theta' x := b \wedge [e \vee (e \oplus y)\Theta(e \oplus)]$ It is clear that Θ' is $L \downarrow b.$ -valued,and isotone in y and antitone in *X*.As L is Distributive lattice therefore $x \wedge [e \vee (e \oplus y)\Theta(e \oplus x)] = 0$

As $x \le b$ then $x \wedge (y\Theta' x) = 0$ And $x \vee (y\Theta' x) = b \wedge [x \vee e \vee (e \oplus y)\Theta(e \oplus x)$

As *L* is Distributive lattice therefore $x \vee (y\Theta' x) = b \wedge (e \vee' y)$ $(b \wedge e) \vee b \wedge y) = y$ Hence $x \vee (y\Theta' x) = y$

3.Pseudovariety and Redical class:

Here we define pseudovariety and Redical class for Distributive lattice as follow:

3.1 Pseudovariety (in Distributive lattice): A non-empty class of finite Distributive lattice closed under divisors and finite direct product is called (in Distributive lattice.)

3.2 Redical class(in Distributive Lattice): A redical class of finite Distributive lattice is a Subclass with the following properties:

I. It is closed under homo-morphic images.

II. If *D* is a Distributive lattice and there are three Normal subgroups which belong to this class , and as the product of these Normal subgroups is also a Normal subgroups. Therefore product of these Normal subgroups also belongs to this class III for each Lattice this class is unique.

We define pseudovariety and Redical class for Ring as follow:

3.3 Pseudovariety (in Ring): A non-empty class of finite Ring closed under divisors And finite direct product is called (in Ring.)

3.4 Redical class(in Ring): A redical class of finite Ring is a subclass with the following properties:I. It is closed under homo-morphic images.II. If *R* is a Ring and there are two Normal subgroups which belong to this class , and as the product of these Normal subgroups is also a Normal subgroups. Therefore product of these Normal subgroups also belongs to this class I. For each Lattice this class is unique.

3.5 Theorem : If R_1 and R_2 are pseudovartites of Ring and let *R* be a finite Ring, then

i. $R \in R_1.R_2 \Rightarrow R/R_{R_1} \in R_2$

ii. $$R_{R_1R_2}/R_{R_1} = \left(R/R_1\right)_{R_2}$$

Proof: Let $R \in R_1.R_2$, and we have to prove $R/R_{R_1} \in R_2$ As R_1 and R_2 are pseudovartites of Ring. Therefore by the definition of pseudovartites $R_1.R_2$ is also a pseudovartites .

Let $R_1.R_2$ =K.and as $R \in R_1.R_2$ then $R_1.R_2$ =k must have normal subgroups. And $K \in R_1$ and $R/K \in R_2$ .But we know that by the definition of Radical $K \subseteq R_{R_1}$,and therefore $R/R_{R_1} \in R_{R_2}$.

Let I_1 and I_2 are two Ideals of *R* and $R \in R_1.R_2$ Suppose that $B_i = (I_i)_{R_1}$ As R_1 pseudovariety then $B_1.B_2 \in R_1$ therefore $B_1B_2 \in R_1.R_2$ Now we will prove $I_1.I_2/B_1B_2 \in R_2$ We have $I_1.I_2/B_1B_2 = (I_1B_2/B_1B_2).(I_2B_1/B_1B_2)$,but (I_1B_2/B_1B_2) is a ideal of $I_1.I_2/B_1B_2$ and a homo-morphic image of $I_1./B_1 \in R_2$ and similarly for other factor. The quotient $I_1.I_2/B_1B_2 \in R_2$,therefore this pseudovarity is a fitting class.

3.5.1 Corollary: F_1 and F_2 are pseudovartites of Field [19] and let *R* be a finite Field, then

$$F \in F_1.F_2 = F/F_{F_1} \in F_2 \qquad F_{F_1}/F_{F_1} = \left(F/F_1\right)_{F_2}$$

Proof: As we know that Field has no proper Ideals (it has only F and {0})Therefore it is trivial.

3.6 Theorem: If *V* be an extension –closed pseudovariety of Distributive lattice *D* containing *Ab*. If *D* is a finite Lattice $a \in D_V$, and $b \in D$,then $\langle a,b\rangle \in V$.

Proof: Suppose that $H = \langle a,b\rangle$ as *H* is a cyclic extension of Normal subgroups N= $H \cap D_V$ [1]. So above will true for Distributive lattice.

Theorem 3.7: *V* be an extension – closed pseudovariety of Distributive lattice *D* containing *Ab* . And if V-radical admits a binary characterization then $D_V = \{a \in D : \forall b \in D, \langle a,b\rangle \in V\}$

Proof : Let *U* is a binary characterization of the *V*-radical and let $a,b \in D$. Consider the Subgroup $H_b = \langle a,b\rangle$ .if $a \in D_V$, then $H_b \in V$ (by above Theorem.) .Now, if $H_b \in V$ $\forall b \in D$, then U(*a*,b)=1 for every $u \in U$. As $U \subseteq (\overline{\Omega}S)^V$. And as U is a characterization therefore $a \in D_V$, which implies that $D_V = \{a \in D : \forall b \in D, \langle a,b\rangle \in V\}$

REFERENCES

1. Atiyah M. F., Macdonald, I. G. | 1969, Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.- London-Don Mills, Ont. ix+128 pp . | 2. Birkhoff ,G.; | 1967, Lattice Theory, 3rd ed., Colloquium Publications,vol. 25, Amer. Math. Soc. | 3. Bogart,K., Freese,R., and Kung, J. | 1990, The Dilworth Theorems, Selected Papers of Robert P. Dilworth, Birkhauser, Basel. | 4. Beachy, J. A. | 1999, Introductory Lectures on Rings and Modules. | Cambridge, England: Cambridge University Press. | 5. P. Crawley and R. P. Dilworth, | 1973,Algebraic Theory of Lattices, Prentice- Hall, Englewood Clifls,New Jersey. | 6. R. Dedekind, | 1897, ¨Uber Zerlegungen von Zahlen durchihre | grosste gemeinsamen Teiler, Festschrift der Herzogl. technische | HochschulezurNaturforscherVersammlung,Braunschweig)Reprintein \Gesammelte mathematische Werke", Vol. 2, | pp. 103- 148, Chelsea, NewYork, 1968. | 7. R. Dedekind | 1968, Gesammelte mathematische Werke, Chelsea, New York. | 8. Ellis, G. | 1993, Rings and Fields. Oxford, England: Oxford University Press. | 9. G.Gratzer, E.T.Schmidt | 1958 Characterization of Relativel Complemented Distributive lattice, math.bme.hu | 10. D. Gorenstein. | 1980, Finite groups. Second Edition. Chelsea Publishing Company, | 11. William J. DeMeo | 2012 7 May ,CongruenceLattices Of Finite Algebras, | arXiv:1204.4305v3 [math.GR] . | 12. R. Freese, | 1976, Planar sublattices of FM(4), Algebra Universalis 6 69-72. | 13. R. Freese, C. Herrmann and A. P. Huhn, | 1981, On some identities valid in modular congruence varieties, Algebra Universalis 12 322-334. | 14. R. Freese and B. Jonsson, | 1976, Congruence modularity implies the Arguesian identity, Algebra Universalis 6 , 225-228. | 15. Ralph Freese and Ralph McKenzie, | 1987, Commutator Theory for Congruence Modular Varieties, London Math. Soc. Lecture Note Series vol. 125, Cambridge University Press, Cambridge. | 16. Giambruno, E. Jespers and A. Valenti, | 1994, Group identities on units of rings, Archiv der Mathematik 63, 291-296. | 17. A. Giambruno, S. K. Sehgal and A. Valenti, | 1977, Group algebras Whose units satisfy a group identity, Proc. AMS (to appear).Interscience, New York. | 18. B. C. Hall, | 2000 31 may an Elementary Introduction to Groups and Representations arxiv:math-ph/0005032v1 . | 19. Nagell, T. | 1951,"Moduls, Rings, and Fields." §6 in Introduction to | Number Theory. New York: Wiley, pp. 19–21. | 20. F. Wehrun2010 16 Aug .A non – coordinatizables sectionally Modular Lattice with a Large Jonsson four frame arXiv:1003.5158v2[math.RA] |