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Mathematical Analysis and Accurate Solving of Four-Center Arcs For Fitting Ellipse Based on Dichotomy

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ABSTRACT

It is a common method in numerical control machining field using arcs to fit ellipse. As many numerical control machines do not support ellipse interpolation, ellipse machining is usually realized by arc machining using fitting method. Thus mismachining tolerance depends on theoretical fitting precision. However, present fitting ellipse is not precise due to inaccurate error algorithm of four-center arcs for machining. This work derives analytical form of normal error and determines finite solution interval of four-center arcs method for fitting ellipse. The transcendental equation for getting the error is derived on the basis of graphics theory, and the equation is solved using dichotomy. The minimum error band of four-center arcs for fitting ellipse has been determined for a given normal error, and this is realized by Visual LISP language for programming under AutoCAD environment. The optimal solution of four-center arcs for fitting ellipse has been determined, and the accurate criterion is supplied to determine whether four-center arc is feasible to fit ellipse for a given form tolerance.

KEYWORDS four-center arcs, arithmetic fitting, ellipse, the optimal solution, dichotomy

1. Introduction

Featuring simple computation and easy realization, four-center arcs for fitting ellipse is free from complicated computation of ellipse fitting, and has been widely utilized in engineering. Ellipse is an ideal curve given in design, but in numerical control, the curve machined by adopting fitting arc as operation control trace of cutting tools always has error, therefore, the only choice is to control ellipse in allowable scope of change, which is defined as ellipse tolerance band. According to stipulations of GB/T1182-1996 in Reference [1] on profile of a line, ellipse form tolerance band shall be two inner/outer equal-space lines of ellipse, the corresponding error shall be calculated along the normal direction of ellipse [Fig.1]. At present, numerous References have discussed the error of ellipse fitting, References [2,3,7,8] have analyzed deviation change rule of four-center arcs for fitting ellipse, the given deviation is along Y-axis direction, the given error is along fitting arc radial direction of ellipse. The errors given in these methods are not completely identical with the normal error actually required to be controlled for ellipse, therefore, it is impossible to describe precision of four-center arcs for fitting ellipse precisely, the degree of approximation of fitting is blur. References [6,9,10] introduce the arc fitting method of listed curve, i.e. to describe the approximate orientation of work piece surface shape curve by discrete offset point of curvilinear coordinates of certain coordinate on given curve, this method is simple, but error equation is not given, and the precision is poor. References [11-15] have researched numerous methods of ellipse and curve fitting in numerical control machining field, the general practice is to determine the center of circle as per profile curve length equivalent to arc length and change rule of curve segment gradient; use curve equivalent arc length arc fitting method replacing convex curve with convex arc, and use series expansion in this method to simplify transcendental triangle equation; so as to avoid the trouble of iterative solution. The disadvantages of these methods are failure to give accurate analytical form of fitting error in theory, therefore, it is impossible to describe fitting precision better.

In order to better apply four-center arcs for fitting ellipse, it is necessary to determine normal error of four-center arcs for fitting ellipse accurately. Finite solution interval of four-center arcs for fitting ellipse and analytical algorithm are determined as per following figure, so as to derive normal error and the minimum error band reachable by corresponding four-center arcs for fitting ellipse, and provide accurate criterion to determine whether four-center arc can fit ellipse of given form tolerance.

2. Finite solution interval of four-center arcs for fitting ellipse

Parametric equation of ellipse is:

$$x = a\cos\alpha,\ y = b\sin\alpha \qquad (1)$$

where, α is parameter angle of ellipse, $0^\circ \le \alpha \le 360^\circ$, a、b are transverse/longitudinal half-axis length, $b > a > 0$, and $a \ne b$; if $a = b$, then it is parametric equation of a circle. As ellipse is symmetrical, therefore, analytical form is analyzed and derived as per ellipse of the first quartile if $a < b$, analytical form can be acquired by symmetrical conversion of results of the first quartile for other quartile, if $a > b$, solve by exchanging a、b, then proceed with symmetrical conversion of results in relation to straight line $x = y$.

As per the setting of $a < b$, in four-center arc of fitting ellipse, the arc radius passing transverse axis extreme point must be larger than the arc radius passing longitudinal axis extreme point, for the sake of discussion, the arc passing transverse axis extreme point is referred to as large arc, and the arc passing longitudinal axis extreme point is referred to as small arc.

In order to approximate ideal ellipse better, in accordance with characteristics of ellipse, arc of fitting ellipse shall normally meet following six conditions at the same time:

(1) Pass long/short axis extreme point of ellipse;
(2) Symmetrical in relation to center of ellipse;
(3) Maintain first derivative continuity;
(4) Within given ellipse tolerance band;
(5) Fitting arcs are tangent to each other;

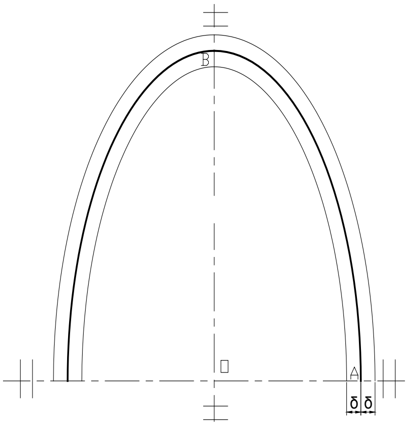


Figure 1: Ellipse tolerance zone (1/2)

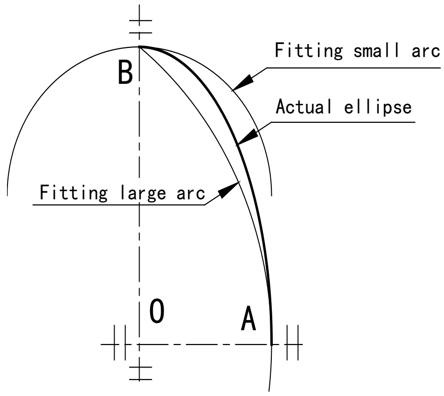


Figure 2: Fitting arc (1/4)

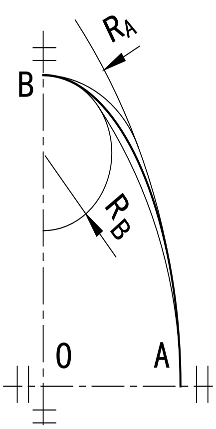


Figure 3: Finite solution space of fitting arc

Finite solution interval of four-center arcs for fitting that meet the first four conditions are as shown in figure 2, where the center of large arc is on transverse axis, its maximum radius is infinite, the minimum fitting arc approximates the arc of which the center is the point of intersection between perpendicular bisector of connection line of two extreme points and transverse axis and passing two extreme points; the center of small arc is on longitudinal axis, its minimum radius goes to 0 and the maximum radius is a.

In order to meet the 5th condition mentioned above, as per setup of $a < b$, as curvature radius [4] R_A of the ellipse transverse axis extreme point must be larger than curvature radius R_B of longitudinal axis extreme point, therefore, arc radius between transverse axis extreme point and longitudinal axis extreme point fitting ellipse must transit from large to small, vice versa, must transit from small to large. Ideal scenario of four-center arcs for fitting ellipse is that maximum value of large arc radius and minimum value of small arc radius take curvature radius of respective extreme point, so that finite solution interval of four-center arcs for fitting ellipse is shortened further, as shown in figure 3, normal error of four-center arc in the interval is determined accordingly, see details as per 4.

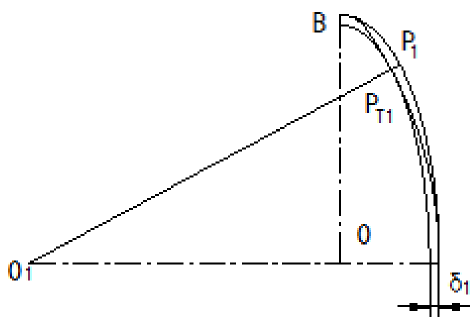


Figure 4: Normal error of large arc

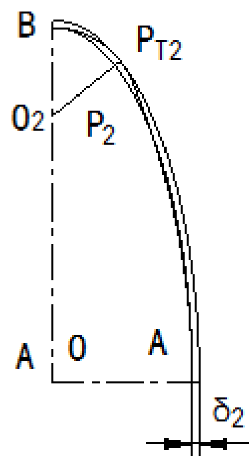


Figure 5: Normal error of small arc

for fitting ellipse

Four-center arcs for fitting can be determined as per ellipse transverse/longitudinal half-axis a,b and nominal error. Analytical form is derived as follows:

Large arc is as shown in figure 4, set its normal error as δ_1 (for the sake of discussion, take $\delta_1 \geq 0$), circle center as O_1, radius as R_1, tangent point between equal-space line of ellipse determined by δ_1 and large arc as P_{T1}, connection line between O_1 and p_{T1} intersects ellipse at P_1, the corresponding ellipse parameter angle is α_1, then the normal gradient of point P_1 on ellipse is equivalent to normal gradient of connection line between O_1 and P_1:

$$Y_1 / \left(X_1 - X_{o1}\right) = a / b \tan\alpha_1$$

Substituting coordinate of point P_1 as per formula (1-1), we have:

$$X_{o1} = \left(a - b^2 / a\right)\cos\alpha_1 \tag{2}$$

According to geometrical relation, it can be derived that:

$$\left(R_1 + \delta_1\right)^2 = \left(X_1 - X_{o1}\right)^2 + Y_1^2 \tag{3}$$

$$R_1 = a - X_{o1} \tag{4}$$

Substituting coordinate of point P_1 as per formula (1) into formula (3) and combining with simultaneous formula (2) and (4), we have:

$$(a^2 - b^2)\cos^2\alpha_1 - 2\left(a + \delta_1\right)\left(a^2 - b^2\right)$$
$$/a\cos\alpha_1 + \left(a + \delta_1\right)^2 - b^2 = 0$$

The equation is a quadratic equation with one unknown of $\cos\alpha_1$, there are two solutions, the discriminant is:

$$\Delta = 4 \times \left[2a + \delta_1\right]\delta_1\left(b^2 - a^2\right)b^2 / a^2$$

As shown above, $a = b$, $b > a > 0$, $\delta_1 \geq 0$, therefore $\Delta \geq 0$, but it is limited only to the first quartile, one solution can be left, so we have:

$$\cos\alpha_1 = \left[a + \delta_1 - \sqrt{\left(2a + \delta_1\right)\delta_1 / \left(b^2 - a^2\right)}\, b\right] / a \tag{5}$$

Small arc is shown in figure 5. Set its normal error as δ_2 (for the sake of discussion, take $\delta_2 \geq 0$ below), circle center as O_2, radius as R_2, tangent point between equal-space line of ellipse determined by δ_2 and small arc is as P_{T2}, connection line between O_2 and P_{T2} intersects ellipse at P_2, the corresponding ellipse parameter angle is α_2, as per the derivation method of transverse axis extreme point, we get:

$$Y_{o2} = \left(b - a^2 / b\right)\sin\alpha_2 \tag{6}$$

$$\left(R_2 - \delta_2\right)^2 = \left(Y_2 - Y_{o2}\right)^2 + X_2^{\,2} \tag{7}$$

$$R_2 = b - Y_{o2} \tag{8}$$

$$\sin\alpha_2 = \left[b - \delta_2 - \sqrt{\left(2b - \delta_2\right)\delta_2 / \left(b^2 - a^2\right)}\, a\right] / b \tag{9}$$

As four-center arcs for fitting of ellipse must be tangent, therefore δ_1 and δ_2 are not independent from each other, the associated solution derivation is as follows:

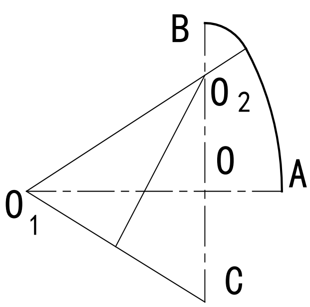


Figure 6: Solution fitting small arc

Assuming that δ_1 is known, determine circle center O_1 and radius R_1 of large arc as per formula (2), (4) and (5), solve small arc tangent with it as per figure 6, take $BC = R_1$, the point of intersection between perpendicular bisector of O_1C and longitudinal point is O_2, the transverse coordinate is 0 and longitudinal coordinate is:

$$Y_{o2} = \left[b - R_1 - \frac{X_{o1}^{\ 2}}{b - R_1} \right] \tag{10}$$

Combining simultaneous formula (10) and formula (6)-(8), we get the radius R_2 of small arc and its error δ_2.

Set central distance of large/small arc $L = \sqrt{x_{o1}^{\ 2} + y_{o2}^{\ 2}}$, then the point of tangency of large/small arc of first quartile fitting ellipse is:

$$X_T = X_{o1}\left(1 - R_1 / L\right), Y_T = Y_{o2} R_1 / L \tag{11}$$

In summary of above mentioned, when transverse/longitudinal half-axis length a and b of ellipse and normal error δ_1 of large arc are given, four-center arcs for fitting can be determined.

Similarly, when transverse/longitudinal half-axis length a and b of ellipse and normal error δ_2 of small arc are given, four-center arcs for fitting can be determined as per the derivation method mentioned above.

4. Error analysis of four-center arcs for fitting ellipse

As per the finite solution interval mentioned in 2, on the above basis, in accordance with transverse/longitudinal half-axis a and b, normal error scope of four-center arcs for fitting ellipse can be derived.

According to the tangent relationship between large/small arc, we have:

$$\left(R_1 - R_2\right)^2 = X_{o1}^{\ 2} + Y_{o2}^{\ 2} \tag{12}$$

Solving simultaneous formula (12) and formula (4)/(8), we have:

$$a^2 + b^2 - 2bR_2 - 2aR_1 + 2R_1R_2 = 0 \tag{13}$$

As shown by geometrical relation, within finite solution interval shown in figure 3, radius of large/small arc increases or decreases simultaneously, with the decrease of fitting arc radius, normal error of large arc increases gradually, and normal error of small arc decreases gradually. Therefore, when radius of small arc R_2 takes the minimum value, i.e. curvature radius of ellipse at the extreme point $R_B = a^2 / b$, substitute formula (13), we get minimum value R_1:

$$R_{1\min} = \left(b + a\right) b / 2a \tag{14}$$

At the moment, horizontal coordinate of circle center of large arc $X_{oA} = a - R_{1\min}$, substitute $X_{O1} = X_{OA}$ into formula (2), we get corresponding ellipse parameter angle cosine:

$$\cos\alpha_A = \left(b + 2a\right) / 2\left(b + a\right) \tag{15}$$

To be determined as per the α_A. Substitute $\alpha = \alpha_A$ into formula (1), substitute the coordinate of point P_1 and $R_1 = R_{1\min}$ into formula (3), the maximum error of large arc in above finite solution interval is:

$$\delta_{1\max} = \sqrt{\left(a\cos\alpha_A - X_{OA}\right)^2 + b^2\left(1 - \cos^2\alpha_A\right)} - R_{1\min} \tag{16}$$

Therefore, error interval of large arc is:

$$0 \le \delta_1 \le \delta_{1\max} \tag{17}$$

When large arc radius R_1 takes the maximum value, i.e. curvature radius of ellipse at the extreme point $R_A = b^2 / a$, substitute into formula (13), we get the maximum value R_2:

$$R_{2\max} = \left(b + a\right) a / 2b \tag{18}$$

At the moment, circle center longitudinal coordinate of small arc $Y_{OB} = b - R_{2\max}$, substitute $Y_{O2} = Y_{OB}$ into formula (6), we get the corresponding ellipse parameter angle sine:

$$\sin\alpha_B = \left(a+2b\right)/2\left(b+a\right) \quad (19)$$

To be determined as per the α_B. Substitute $\alpha = \alpha_B$ into formula (1), substitute the coordinate of point P_2 and $R_2 = R_{2\max}$ into formula (7), the maximum error of small arc in above finite solution interval is:

$$\delta_{2\max} = R_{2\max} - \sqrt{a^2\left(1-\sin^2\alpha_B\right)+\left(b\sin\alpha_B - y_{OB}\right)^2} \quad (20)$$

Therefore, error interval of small arc is:

$$0 \le \delta_2 \le \delta_{2\max} \quad (21)$$

The relationship between normal error δ_2、δ_1 of large/small arc within interval of formula (17) and (21) plotted as per analytical form derived as per 2 is as shown in Figure 7. As shown in the figure, there are three scenarios of normal error of four-center arc: at point δ_0, $\delta_1 = \delta_2 = \delta_0$; where δ_1 is smaller than δ_0, $\delta_2 > \delta_0 > \delta_1$; where δ_1 is larger than δ_0 , $\delta_2 < \delta_0 < \delta_1$.

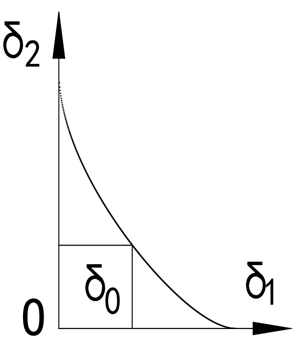


Figure 7: Solving fitting small arc of normal error δ_2 and δ_1

5. Transcendental equation of four-center arcs for fitting ellipse error

The optimal solution of four-center arcs for fitting of ellipse can be determined as per its long/short half-axis. Derivation of analytical form is as follows:

Substitute formula (4) and (8) into formula (12), we get:

$$\left(b-a\right)^2 - 2\left(b-a\right)\left(Y_{O2} - X_{O1}\right) - 2Y_{O2}X_{O1} = 0 \quad (22)$$

As mentioned in Section 1, four-center arc of fitting ellipse has countless solutions. When normal error of ellipse extreme point fitting arc is equivalent, i.e. $\delta_1 = \delta_2 = \delta_0$, we get the followings by solving simultaneous formula (2), (5), (6), (9) and (22):

$$1-2\left(b+a\right)\{[b-\delta_0 - \sqrt{\left(2b-\delta_0\right)\delta_0/\left(b^2-a^2\right)}a]/b^2 + \left[a+\delta_0 - \sqrt{\left(2a+\delta_0\right)\delta_0/\left(b^2-a^2\right)}b\right]/a^2\} + 2\left[b-\delta_0-\sqrt{\left(2b-\delta_0\right)\delta_0\left(b^2-a^2\right)}a\right]/b^2 \times \left[a+\delta_0-\sqrt{\left(2a+\delta_0\right)\delta_0/\left(b^2-a^2\right)}b\right]/a^2 \times \left(b+a\right)^2 = 0 \quad (23)$$

This is a transcendental equation for δ_0 and it is impossible to be solved directly. Therefore, we set

$$f\left(\delta_0\right) = 1-2\left(b+a\right)\{[b-\delta_0 - \sqrt{\left(2b-\delta_0\right)\delta_0/\left(b^2-a^2\right)}a]/b^2 + \left[a+\delta_0 - \sqrt{\left(2a+\delta_0\right)\delta_0/\left(b^2-a^2\right)}b\right]/a^2\} + 2\left[b-\delta_0-\sqrt{\left(2b-\delta_0\right)\delta_0\left(b^2-a^2\right)}a\right]/b^2 \times \left[a+\delta_0-\sqrt{\left(2a+\delta_0\right)\delta_0/\left(b^2-a^2\right)}b\right]/a^2 \times \left(b+a\right)^2$$

Plotting curve of $f\left(\delta_0\right)$ with changing δ_0, one gets figure 8.It is seen that $f\left(\delta_0\right) > 0$ as $\delta_0 \to 0$; While $\delta_0 \to \min(\delta_{1\max}, \delta_{2\max})$, $f\left(\delta_0\right) < 0$. Since $f\left(\delta_0\right)$ monotonously decreases with increasing δ_0 , it is feasible to solve equation (23) and get accurate δ_0 using dichotomy within the interval of $\left(0, \min\left(\delta_{1\max}, \delta_{2\max}\right)\right]$. The accurate δ_0 was obtained by Visual LISP programming under Auto CAD environment。The error band of four-center arc for fitting ellipse corresponding to δ_0 is the area between the parallel lines as shown in Fig. 9(a). In the case of $\delta_2 > \delta_1$, the fitting arc is beyond the error band from the outer side as shown in Fig. 9(b). In the case of $\delta_2 < \delta_1$, the fitting arc is beyond the error band from the inner side, see Fig. 9(c). Therefore, the error band determined by δ_0 is the smallest one among those of four-center arc for fitting ellipse. The fitted ellipse has the best precision by applying δ_0 to the four-center arc fitting. It is impossible to get such precision by either considering Y coordination or radial calculation of arcs.

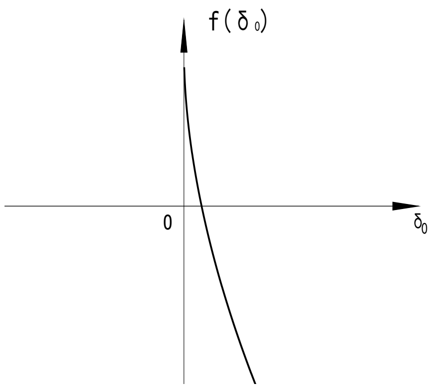
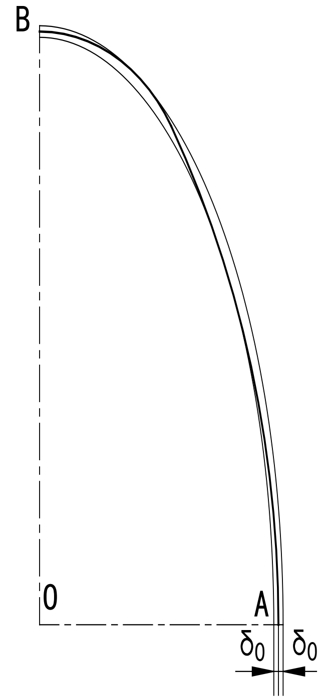
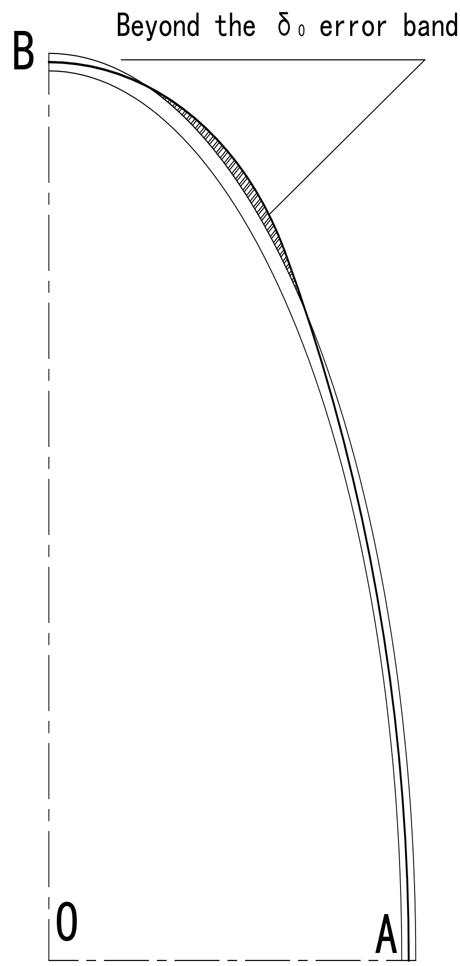


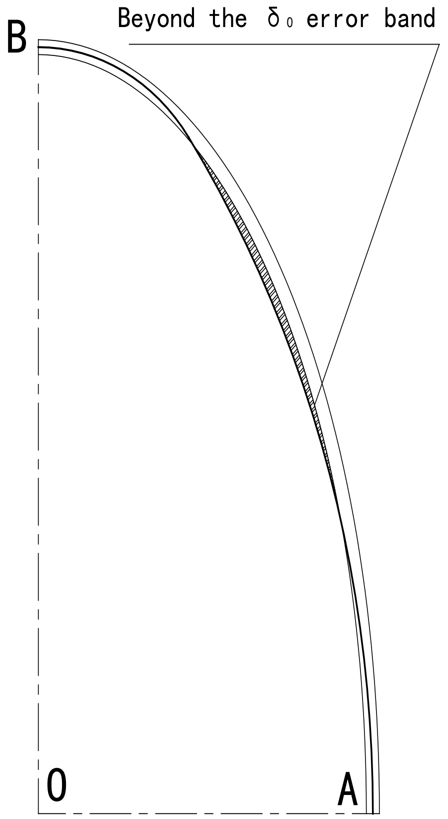
Figure8: $f\left(\delta_0\right)$ curve with varying δ_0



(a) $\delta_2=\delta_1=\delta_0$



(b) $\delta_2>\delta_1$



(c) $\delta_2<\delta_1$

Figuer9: Fitting arcs with different normal errors.

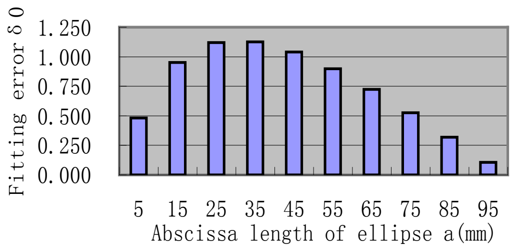


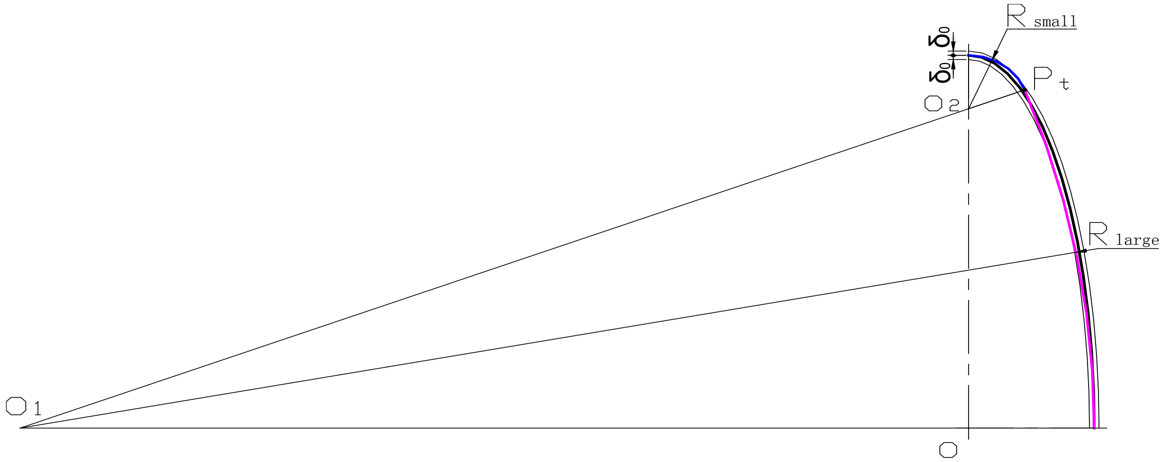
Figure10: Normal error of the fitted ellipses with different length of abscissa axis.

The normal error of four-center arc is obtained by programming and plotted in Fig. 10, following the aforementioned emthod, setting the long axis b as 100，the short axis a varying from 5 to 9. Parameters of the ellipse and the simulated results are displayed in Table 1. According to Fig. 10, the fitting error δ_0 changes with the axis of the ellipse. δ_0 has the maximum value if the short axis is smaller than one half of the long axis. δ_0 tends to be zero while the length of the short axis is close to that of the long axis or to zero. It is indicated that the best precision of the normal error by four-center arc fitting ellipse depends on the length of the axis.

Fig. 11 illustrates the results of the ellipse with a 30 and b 100.

Table 1: Ellipse parameters and fitting results

a	b	δ_0	Fitting small arc			Fitting large arc			Coordination of tangent point	
			x_{o1}	y_{o1}	R_1	x_{o2}	y_{o2}	R_2	x_{P_t}	y_{P_t}
5	100	0.482	0	98.779	1294.130	-1289.130	0	1.221	1.218	98.872
15	100	0.952	0	94.777	469.475	-454.475	0	5.223	5.113	95.844
25	100	1.119	0	89.090	299.614	-274.614	0	10.910	10.378	92.457
35	100	1.126	0	81.855	225.337	-190.337	0	18.146	16.669	89.023
45	100	1.041	0	73.137	183.393	-138.393	0	26.863	23.750	85.689
55	100	0.899	0	62.977	156.322	-101.322	0	37.023	31.444	82.521
65	100	0.723	0	51.402	137.344	-72.344	0	48.598	39.617	79.550
75	100	0.527	0	38.431	123.267	-48.267	0	61.569	48.166	76.782
85	100	0.320	0	24.082	112.387	-27.387	0	75.918	57.012	74.213
95	100	0.107	0	8.367	103.713	-8.713	0	91.634	66.096	71.834



Figuer11: fitting results drawing

6. Conclusion

Numerous solutions can be obtained by four-center arc fitting ellipse. The transcendental equation of the four-center arc fitting ellipse has been theoretically derived for normal error and precisely solved based on dichotomy. When the five boundary conditions are satisfied, an area between two parallel lines that denotes the smallest normal error can be easily determined according to different length of long and short axis. Thus the optimal solution of four-center arc fitting ellipse can be achieved. The fitted ellipse has the best accuracy, showing a smallest error. Four-center arc fitting ellipse method works well when the absolute value of the symmetrical normal error is no less than the minimum of the normal error. For a given normal error, one can use this method to determine whether four-center arc fitting ellipse method does work or not. In case ellipse with even more precision is fitted, four-center arc method is impropriate. Instead, more arcs are required for the fitting.

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