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Numerical Solution of First-Order Fuzzy Initial Value Problem by Non-Linear Trapezoidal Formulae Based on Variety of Means

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ABSTRACT

In this paper the study of numerical algorithms for solving first-order fuzzy initial value problems based on Seikkala derivative of fuzzy process are considered. The numerical methods discussed are the non-linear Trapezoidal formulae based on variety of means. Sufficient conditions for stability and convergence of the proposed algorithms are given. The accuracy and efficiency of the proposed methods are illustrated by solving a fuzzy initial value problem using triangular fuzzy number.

KEYWORDS Fuzzy differential equation, non-linear trapezoidal formulae, variety of means, Triangular fuzzy number.

1. Introduction

Fuzzy differential equations are a natural way to model dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations, which appear in many applications. In the recent years, the topic of FDEs has been investigated extensively. The concept of a fuzzy derivative was first introduced by S.L. Chang and L.A. Zadeh in [5], later D. Dubois and H. Prade in [7], who defined and used extension principle and was followed by Puri and Ralescu in [27]. The fuzzy differential equation and the initial value problem were regularly treated by O.Kaleva in [16, 17] and by S. Seikala in [28]. It is difficult to obtain an exact solution for fuzzy differential equations and hence the numerical method for solving fuzzy differential equations is introduced by M.Ma, M. Friedman, A. Kandel in [21] by standard Euler method, by the authors Abbasbandy and Allahviranloo in [2] by Taylor method and by the authors in [1, 14] by Runge-Kutta method. Kanagarajan and Sampath [18,19] developed a numerical algorithm for solving fuzzy differential equations by using Runge-Kutta method and Runge-Kutta Nystrom method of order three. K. Ivaz et.al.[15] derived and implemented a numerical method for fuzzy differential equations and hybrid fuzzy differential equations

Non-linear trapezoidal formulae based on variety of means have found wide applications in the studies like mechanical vibrations, electrical circuits, planetary motions, etc. Several Researchers have applied non-linear trapezoidal formulae based on variety of means and analyze their error control and stability analysis.
NECDET BILDIK and MUSTAFA INC [23] studied on the numerical solution of Initial value problems for Non-linear trapezoidal formulae with different types. D. J .Evans and B.B.Sanugi [8, 9] compared the non-linear trapezoidal formulae using Euler, Harmonic mean, and logarithmic mean for solving IVPs ABDUL-MAJID WASWAS [29] made a comparison of modified Runge - kutta formulae based on variety of means. K. Murugesan ,et.al.[22] have done a comparison of extended Runge-kutta formulae based on variety of means to solve systems of IVPs.

In engineering and physical problems, Trapezoidal rule is a simple and powerful method to solve numerically related ODEs. Trapezoidal rule has a higher convergence order in comparison to other one step methods, for instance, Euler method.

In this work, we concentrate on numerical procedure for solving FDEs, whenever these equations possess unique fuzzy solutions.

In Section 2, we briefly present the basic definitions, Trapezoidal rule for solving ordinary differential equations and non-linear trapezoidal formulae based on variety of means are given. A fuzzy initial value problem is defined in Section 3. Trapezoidal rule for solving FIVP, convergence and stability of the mentioned method are proved and non-linear trapezoidal formulae based on variety of means are given in section 4. The proposed algorithm is illustrated by solving an FIVP in Section 5.

2. Preliminary notes

2.1 NON-LINEAR TRAPEZOIDAL FORMULAE

2.1.1 Increment Function of Different Means:

We consider the initial value problem for a first order ordinary differential equation

$y' = f(t,y), y(t_0) = y_0$

(2.1)

The function $f(t,y)$ is to be continuous for all (t,y)in some domain D of the xy-plane , and (x_0, y_0) is a point in D where D={(t , y) / a ≤ t ≤ b, -∞ < y < ∞}

Theorem: 2.1 Let $f(t,y)$ be a continuous function of t,y, for all (x,y)in D, and let (x_0, y_0)be an interior point of D. Assume $f(x,y)$satisfies the Lipschitz condition $|f(t,y_1) - f(t,y_2)| \leq K|y_1 - y_2|$ all (t, y_1), (t, y_2) in D for some K ≥ 0. Then for a suitably chosen interval I=[x_0-α, x_0+α], there is a unique solution y(x) on I of (2.1)

Suppose we want to compute an approximate value $y(t)$ (t being some real number) which will be existent and unique. To compute this numerical approximation, we use one-step method of the form $y_{n+1} = y_n + h\Phi(t_n, t_{n+1}, y_n, y_{n+1}, h), n = 0,1, \ldots k-1.$
(2.2)

with an increment function Φ of certain special type. Here, n is a (user-chosen) arbitrary step size parameter, and put $h = \frac{t_n - t_0}{n}$ and $t_n = t_0 + nh, for\ n = 1, \ldots, k.$ (2.3)
Finally, we take the value $y(t{:}h) = y_n$ as an approximation of the desired result $y(t)$.
One of the most popular choices for the increment function Φ is the arithmetic mean of the two values $f_n = f(t_n, y_n)$ and $f_{n+1} = f(t_{n+1}, y_{n+1})$
i.e. $\Phi^A(t_n, t_{n+1}, y_n, y_{n+1}, h) = \frac{f_n + f_{n+1}}{2}$
which yields the **trapezoidal rule**.

However, it has been pointed out by D. J. Evans and B.B. Sanugi that taking the arithmetic mean is not in all cases the best choice, and consequently the use of other means, e.g. Geometric, Harmonic or logarithmic ones has been studied. The resulting methods were denoted as **non-linear trapezoidal formulae.**

Now, let Φ denote one of the following increment functions, defined through different types of non- linear means:
$\Phi^G(t_n, t_{n+1}, y_n, y_{n+1}, h)$=$\sqrt{f_n f_{n+1}}$(Geometric)
$\Phi^{Ha}(t_n, t_{n+1}, y_n, y_{n+1}, h)$=$2\left(\frac{f_n f_{n+1}}{f_n + f_{n+1}}\right)$
(Harmonic)
$\Phi^L(t_n, t_{n+1}, y_n, y_{n+1}, h)$= $\left(\frac{f_{n+1} - f_n}{In\left(\frac{f_{n+1}}{f_n}\right)}\right)$
(Logarithmic)
$\Phi^{Co}(t_n, t_{n+1}, y_n, y_{n+1}, h)$=$\left(\frac{f_n^2 + f_{n+1}^2}{f_n + f_{n+1}}\right)$
(Contra-Harmonic)
$\Phi^{He}(t_n, t_{n+1}, y_n, y_{n+1}, h)$=$\frac{1}{3}\left(f_n + f_{n+1} + \sqrt{f_n f_{n+1}}\right)$ (Heronian mean)
$\Phi^{Ce}(t_n, t_{n+1}, y_n, y_{n+1}, h)$=$\frac{2}{3}\left(\frac{f_n^2 + f_n f_{n+1} + f_{n+1}^2}{f_n + f_{n+1}}\right)$
(Centroidal Mean)
$\Phi^R(t_n, t_{n+1}, y_n, y_{n+1}, h)$=$\sqrt{\frac{f_n^2 + f_{n+1}^2}{2}}$ (Root mean square)

Furthermore, that each of the increment functions defines a one-step method of order 2, as long as certain normality conditions on f are satisfied.

The standard Trapezoidal formula based on arithmetic mean (AM) formula for solving initial value problems of the form $y' = f(t,y), y(t_0) = y_0$ is written in the form $y_{n+1} = y_n + h\left(\frac{f_n + f_{n+1}}{2}\right)$, where h is the mean length in the local t-direction.

The Non-linear Trapezoidal formulae based on variety of means and their Local Truncation Errors, and Stabilities are calculated and listed in Table 2.1.

Table 2.1: Non-linear trapezoidal formulae

Sl. No	Mean	Notation	Formula $y_{n+1}=$	LTE $y(t_{n+1})-y_{n+1}$	Stabiliy
1	Arithmetic Mean	AM	$y_n+h\left(\frac{f_n+f_{n+1}}{2}\right)$	$h^3\left(-\frac{y_n'''}{12}\right)+o(h^4)$	A-Stable
2	Geometric Mean	GM	$y_n+h\sqrt{f_nf_{n+1}}$	$h^3\left[-\frac{1}{12}y_n'''+\frac{1}{8}\frac{(y_n'')^2}{y_n'}\right]+o(h^4)$	L-Stable
3	Harmonic Mean	HaM	y_n +2h $\left(\frac{f_nf_{n+1}}{f_n+f_{n+1}}\right)$	$h^3\left(\frac{-y_n'''}{12}+\frac{y_n''^2}{4y_n'}\right)+o(h^4)$	A-Stable
4	Heronian Mean	HeM	$y_n+\frac{h}{3}\left(f_n+f_{n+1}+\sqrt{f_nf_{n+1}}\right)$	$h^3\left(\frac{-y_n'''}{12}+\frac{y_n''^2}{24y_n'}\right)+o(h^4)$	A-Stable
5	Contraharmonic Mean	CoM	$y_n+h\left(\frac{f_n^2+f_{n+1}^2}{f_n+f_{n+1}}\right)$	$h^3\left(\frac{-y_n'''}{12}+\frac{y_n''^2}{4y_n'}\right)+o(h^4)$	A-Stable
6	Root Mean Square	RMS	$y_n+h\sqrt{\frac{f_n^2+f_{n+1}^2}{2}}$	$h^3\left(\frac{-y_n'''}{12}-\frac{y_n''^2}{8y_n'}\right)+o(h^4)$	A-Stable
7	Centroidal Mean	CeM	$y_n+\frac{2h}{3}\left(\frac{f_n^2+f_nf_{n+1}+f_{n+1}^2}{f_n+f_{n+1}}\right)$	$h^3\left(\frac{-y_n'''}{12}-\frac{y_n''^2}{12y_n'}\right)+o(h^4)$	A-Stable
8	Logarithmic Mean	LM	$y_n+h\left(\frac{f_{n+1}-f_n}{In\left(\frac{f_{n+1}}{f_n}\right)}\right)$	$h^3\left(\frac{-y_n'''}{12}-\frac{y_n''^2}{12y_n'}\right)+o(h^4)$	L-Stable

2.2 TRAPEZOIDAL RULE FOR O.D.E:

The introduction of the most basic definition of differential equations (ODEs) calculus are given as follows.

Consider the first-order differential equation

$$y'(t)=f\big(t,y(t)\big),\qquad y(t_0)=y_0 \tag{2.4}$$

where $f:[t_0,t_N]\times\mathbb{R}^n\to\mathbb{R}^n$ *and* $t_0\epsilon\,\mathbb{R}$. A linear multistep method applied to (2.4) is

$$\sum_{i=0}^{k}\alpha_i y_{m+i}=h\sum_{i=0}^{k}\beta_i f(t_{m+i},y_{m+i})$$

and the especial case of implicit methods, m=1,$\alpha_0=-1,\alpha_1=1$ *and* $\beta_0=\beta_2=1/2$, corresponds to the Trapezoidal rule:

$$y_{m+1}=y_m+\frac{h}{2}[f(t_m,y_m),f(t_{m+1},y_{m+1})] \tag{2.7}$$

for an explicit method, the current value y_{m+k} directly in terms of $y_{m+j},f_{m+j},$ $j=0,1,\dots,k-1,$ which, at this stage of the computation, have already been calculated. An implicit method will call for the solution, at each stage of computation, of the equation

(2.5)

with $\alpha_i, \beta_i \in \mathbb{R}, \alpha_k \neq 0$, given starting discretizations with respect to the grid points

$t_m = t_0 + mh$. The value y_{m+i} is an approximation of the exact solution at t_{m+i}. The special case of explicit methods, m=2, $\alpha_0 = -1,\ \alpha_1 = 0,\ \alpha_2 = 1,\ \beta_0 = \beta_2 = 0$,and $\beta_1 = 2$, corresponds to the Midpoint rule:

$$y_{m+2} = y_m + 2hf(t_{m+1}, y_{m+1})$$

(2.6)

Associated with the multistep method (2.5), we define the first characteristic polynomial as follows:

$$\rho(\xi) := \sum_{i=0}^{k} \alpha_i \xi^i$$

(2.10)

Theorem 2.2

A multistep method is stable if the first characteristic polynomial satisfies the root condition, that is, the roots of $\rho(\xi)$ lie on or within the unit circle, and further the roots on the unit circle are simple.

According to Theorem 2.2, we know the Midpoint rule and Trapezoidal rule are stable.

$$\varrho[y(t); h] = \sum_{j=0}^{k} [\alpha_j y(t + jh) - h\beta_j y(t + jh)]$$

(2.11)

Definition 2.2

The difference operator and the associated linear multistep method (2.5) are said to be of order p if for the following equation:

$$\varrho[y(t); h] = C_0 y(t) + C_1 h y^{(1)}(t) + \ldots + C_q h^{(q)} y^{(q)}(t) + \ldots,$$

(2.12)

$$y_{m+k} = h\beta_m f(t_{m+k}, y_{m+k}) + g$$

(2.8)

where g is a known function of previously calculated value $y_{m+j}, f_{m+j}, j = 0,1, \ldots k - 1$. when the original differential equation in (2.4) is linear, then (2.8) is linear in y_{m+k}, and there is no problem in solving it. When f is nonlinear, for finding solution, we can use the following iteration:

$$y_{m+k}^{[s+1]} = h\beta_k f\left(t_{m+k}, y_{m+k}^{[s]}\right) + g$$

(2.9)

Definition 2.1.

We have $C_0 = C_1 = \ldots = C_p = 0, C_{p+1} \neq 0$,where $C_0 = \sum_{j=0}^{k} \alpha_j$ *and*

$$C_i = (1/i!)\left(\sum_{j=0}^{k} \alpha_j j^i - i\sum_{j=0}^{k} \beta_j j^{i-1}\right), for\ i \geq 1.$$

According to Definition 2.2 of Trapezoidal rule is second-order method. We now recall some general concepts of fuzzy set theory.

Definition 2.3

Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0, 1]$, and $u(x)$ is interpreted as the degree of membership of an element x in fuzzy set u for each $x \in X$.

Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis, that is,

$$u : \mathbb{R} \to [0, 1],$$

(2.13)

Satisfying the following properties:

(i) u is normal, that is, there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$.

(ii) u is a convex fuzzy set $(i.e., u(ts + (1 - t)r) \geq min\{u(s), u(r)\}, \forall t \in [0, 1], s, r \in \mathbb{R}$

(iii) u is upper semicontinuous on $\mathbb{R}$.

(iv) $cl\{s \in \mathbb{R} \mid u(s) > 0\}$ is compact, where cl denotes the closure of a subset. The space $\mathbb{R}_F$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_F$. For $0 < \alpha \leq 1$, we denote

$$[u]^\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\} \quad (2.14)$$

$$[u]^0 = cl\{s \in \mathbb{R} \mid u(s) > 0\}$$

Then from (i) – (iv), it follows that the α – level set $[u]^\alpha$ is a nonempty compact interval for all $0 \leq \alpha \leq 1$, The notation

$$[u]^\alpha = \left[\underline{u}^\alpha, \overline{u}^\alpha\right] \quad (2.15)$$

denotes explicitly the α-level set of u.

The following remark shows when $\left[\underline{u}^\alpha, \overline{u}^\alpha\right]$ is a valid α-level set.

Remark 2.1

The sufficient conditions for $\left[\underline{u}^\alpha, \overline{u}^\alpha\right]$ to define the parametric form of a fuzzy number are as follows:

(i) $\underline{u}^\alpha$ is a bounded monotonic increasing (non decreasing) left-continuous function on (0, 1] and right-continuous for $\alpha = 0$.

(ii) $\overline{u}^\alpha$ is a bounded monotonic decreasing (non increasing) left-continuous function (0,1] and right-continuous for $\alpha = 0$.

(iii) $\underline{u}^\alpha \leq \overline{u}^\alpha, 0 \leq \alpha \leq 1$.

For $u, v \in \mathbb{R}_F$ *and* $\lambda \in \mathbb{R}$, the sum $u + v$ and product λu are defined by

$[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda[u]^\alpha, \forall\, \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$, and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}, \text{ by} \quad (2.16)$$

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}^\alpha - \underline{v}^\alpha\right|, \left|\overline{u}^\alpha - \overline{v}^\alpha\right|\right\}, \quad (2.17)$$

The following properties are well known:

$$D(u+w, v+w) = D(u,v), \forall\, u, v, w \in \mathbb{R}_F,$$

$$D(ku, kv) = |k| D(u,v), \forall\, k \in \mathbb{R}, u, v \in \mathbb{R}_F,$$

$D(u+v, w+e) \leq D(u,e), \forall\, u,v,w,e \in \mathbb{R}_F$ and $(\mathbb{R}_F, D)$ is complete metric spaces.

Let I be a real interval. A mapping $y : I \rightarrow \mathbb{R}_F$ is called a fuzzy process and its α-level set denoted by

$$[y(t)]^\alpha = \left[\underline{y}^\alpha(t), \overline{y}^\alpha(t)\right],\ t \in I, \alpha \in (0,1] \quad (2.18)$$

A triangular fuzzy number N is defined by an ordered triple $(x^l, x^c, x^r) \in \mathbb{R}^3$ with $x^l \leq x^c \leq x^r$, where the graph of N(s) is a triangle with base on the interval $[x^l, x^r]$ and vertex at $s = x^c$. An α-level of N is always a closed, bounded interval. We write $N = (x^l, x^c, x^r)$; then

$$[N]^\alpha = [x^c - (1-\alpha)(x^c - x^l),\ x^c + (1-\alpha)(x^r - x^c)], \text{ for any } 0 \leq \alpha \leq 1 \quad (2.19)$$

Definition 2.4

Let $[a,b] \subset I$. The fuzzy integral $\int_a^b y(t)dt$ is defined by

$$\left[\int_a^b y(t)dt\right]^\alpha = \left[\int_a^b \underline{y}^\alpha(t)dt, \int_a^b \overline{y}^\alpha(t)dt\right] \quad (2.20)$$

Provided the Lebesgue integral on the right exist.

Remark 2.2.

Let $[a,b] \subset I$. If $F : I \rightarrow \mathbb{R}_F$ is seikkala differentiable and its Seikkala derivative F is integrable over [a, b], then

$$F(t) = F(t_0) + \int_{t_0}^{t} F(s)ds, \quad (2.21)$$

for all values of t_0, t, where $a \leq t_0 \leq t \leq b$.

Theorem 2.3

Let $(t_i, u_i), i = 0,1, \ldots, n$, be the observed data, and suppose that each of the

$u_i = \left(u_i^l, u_i^c, u_i^r\right)$ is a triangular fuzzy number. Then for each $t \in [t_0, t_n]$, the fuzzy polynomial interpolation is a fuzzy-value continuous function $f : \mathbb{R} \rightarrow \mathbb{R}_F$, where

$$f(t_i) = u_i, f(t) = (f^l(t), f^c(t), f^r(t)) \in \mathbb{R}_F,$$

$$f^l(t) = \sum_{L_i(t) \geq 0} L_i(t)u_i^l\ ,\ \sum_{L_i(t) < 0} L_i(t)u_i^r$$

$$f^c(t) = \sum_{i=0}^{n} L_i(t)u_i^c, \quad (2.22)$$

$$f^r(t) = \sum_{L_i(t)\geq 0} L_i(t)u_i^r \ , \sum_{L_i(t)<0} L_i(t)u_i^l$$

such that and

$$L_i(t) = \prod_{i\neq j}\left((t-t_j)/(t_i-t_j)\right).$$

3.1 Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$y'(t) = f(t, y(t))\,;0 \leq t \leq T$$

$$y(0) = y_0, \qquad (3.1)$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y , y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number.

We denote the fuzzy function y by $y = [y_1, y_2]$. It means that the r-level set of y (t) for $t \in [t_0, T]$ is

$$[y(t)]_r = [y_1(t;r), y_2(t;r)],$$

$$[y(t_0)]_r = [y_1(t_0;r), y_2(t_0;r)], r\varepsilon(0,1],$$

we write $f(t, y) = [f_1(t, y), f_2(t, y)]$ and

$f_1(t, y) = F[t, y_1, y_2]$,

$f_2(t, y) = G[t, y_1, y_2]$.

Because of $y' = f(t, y)$ we have

$f_1(t, y(t); r) = F[t, y_1(t; r), y_2(t; r)]$

$f_2(t, y(t); r) = G[t, y_1(t; r), y_2(t; r)]$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \setminus s = f(t,\tau)\},\ s \in R$$

so fuzzy number $f(t, y(t))$. From this it follows that

$[f(t, y(t))]_r = [f_1(t, y(t); r), f_2(t, y(t); r)]$, $r \in (0,1]$,

where

$f_1(t, y(t); r) = \min\{f(t, u) | u \in [y(t)]_r\}$

$f_2(t, y(t); r) = \max\{f(t, u) | u \in [y(t)]_r\}$.

Theorem 3.1. Let f satisfy

$$\left|f(t,\ v) - f(t,\ \bar{v})\right| \leq g(t,\ \left|v-\bar{v}\right|),\ t \geq 0,\ v,\ \bar{v} \in R.$$

where g : $R_+ \times R_+ \to R_+$ is a continuous mapping such that r → g(t, r) is non decreasing and the initial value problem

$$u'(t) = g(t,\ u(t)),\ u(0) = u_0. \qquad (3.2)$$

has a solution on R_+ for $u_0 > 0$ and that u(t) ≡ 0 is the only solution of (3.2) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique solution.

In most cases analytical solutions may not be found, and a numerical approach must be considered. Some numerical methods such as the fuzzy Euler method, Nystrom method, and predictor-corrector method are there to solve FDEs. In the following, we present some new methods to numerical solution of FDE.

4.1 Trapezoidal Rule for Fuzzy Differential Equations

In the interval $I = [t_0, T]$, we consider a set of discrete equally spaced grid points

$t_0 < t_1 < t_2 < \ldots < t_N = T$. The exact and approximate solutions at $t_n, 0 \leq n \leq N$,

are denoted by $[y(t_n)]^\alpha = \left[\underline{y}^\alpha(t_n)\right.,$ $\left.\overline{y}^\alpha(t_n)\right]$ and $[y_n]^\alpha = \left[\underline{y}_n^\alpha\ ,\ \overline{y}_n^\alpha\right]$, respectively. The grid points at which the solution is calculated are

$$t_n = t_0 + nh, \quad h = \frac{T-t_0}{N}, \ 0 \le n \le N$$

(4.1)

Let $y_p = \left[\underline{y}, \overline{y}\right], 0 \le p < N$ which $f\left(t_p, y_p\right)$ is triangular fuzzy number. We have

$$y\left(t_{p+1}\right) = y\left(t_p\right) + \int_{t_p}^{t_{p+1}} f(t, y(t))\, dt,$$

(4.2)

By fuzzy interpolation, Theorem 2.3, we get,

$$f_1^l(t, y(t)) = l_0(t) f^l\left(t_p, y_p\right) + l_1(t) f^l\left(t_{p+1}, y_{p+1}\right)$$

(4.3)

$$f_1^c(t, y(t)) = l_0(t) f^c\left(t_p, y_p\right) + l_1(t) f^c\left(t_{p+1}, y_{p+1}\right)$$

(4.4)

$$f_1^r(t, y(t)) = l_0(t) f^r\left(t_p, y_p\right) + l_1(t) f^r\left(t_{p+1}, y_{p+1}\right)$$

(4.5)

We get

$f_1(t, y(t)) = \left(f_l^l(t, y(t)), f_l^c(t, y(t)), f_l^r(t, y(t))\right)$,

interpolates $f(t, y(t))$ with the interpolation data given by the value $f\left(t_p, y_p\right)$, and

$l_0(t) = \left(t - t_{p+1}\right)/\left(t_p - t_{p+1}\right)$,

$l_1(t) = \left(t - t_p\right)/\left(t_{p+1} - t_p\right)$.

$$l_0(t) = \frac{t-t_{p+1}}{t_p - t_{p+1}} \ge 0, \ l_1(t) = \frac{t-t_p}{t_{p+1}-t_p} \ge 0$$

(4.6)

From (2.19) for $t_p \le t \le t_{p+1}$ we have and (4.2) it follows that

$$\left[y\left(t_{p+1}\right)\right]^\alpha = \left[\underline{y}^\alpha\left(t_{p+1}\right), \overline{y}^\alpha\left(t_{p+1}\right)\right]$$

(4.7)

where

$$\underline{y}^\alpha\left(t_{p+1}\right) = \underline{y}^\alpha\left(t_p\right) + \int_{t_p}^{t_{p+1}} \{\alpha f^c\left(t, y(t)\right) + (1-\alpha) f^l\left(t, y(t)\right)\} dt$$

(4.8)

$$\overline{y}^\alpha\left(t_{p+1}\right) = \overline{y}^\alpha\left(t_p\right) + \int_{t_p}^{t_{p-1}} \{\alpha f^c\left(t, y(t)\right) + (1-\alpha) f^r\left(t, y(t)\right)\} dt$$

(4.9)

According to (4.2), if (4.3) and (4.4) are situated in (4.8),(4.3) and (4.4) in (4.9), we obtain

$$\underline{y}_{p+1}^\alpha = \underline{y}_p^\alpha + \int_{t_p}^{t_{p-1}} \{\alpha\left[l_0(t) f^c\left(t_p, y_p\right) + l_1(t) f^c\left(t_{p+1}, y_{p+1}\right)\right]\}$$

$$\{+(1-\alpha) \times \left[l_0(t) f^l\left(t_p, y_p\right) + l_1(t) f^l\left(t_{p+1}, y_{p+1}\right)\right]\} dt$$

(4.10)

By integration we have

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_p^{\alpha} + \frac{h}{2} \times \left[\alpha f^c(t_p, y_p) + (1-\alpha) f^l(t_p, y_p) + \alpha f^c(t_{p+1}, y_{p+1}) + (1-\alpha) f^l(t_{p+1}, y_{p+1})\right]$$

By (2.19) deduce

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_p^{\alpha} + \frac{h}{2}\left[\underline{f}^{\alpha}(t_p, y_p) + \underline{f}^{\alpha}(t_{p+1}, y_{p+1})\right] \quad (4.11)$$

Similarly we obtain

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_p^{\alpha} + \frac{h}{2}\left[\overline{f}^{\alpha}(t_p, y_p) + \overline{f}^{\alpha}(t_{p+1}, y_{p+1})\right] \quad (4.12)$$

Therefore, Trapezoidal rule is obtained as

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_p^{\alpha} + \frac{h}{2}\left[\underline{f}^{\alpha}(t_p, y_p) + \underline{f}^{\alpha}(t_{p+1}, y_{p+1})\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_p^{\alpha} + \frac{h}{2}\left[\overline{f}^{\alpha}(t_p, y_p) + \overline{f}^{\alpha}(t_{p+1}, y_{p+1})\right] \quad (4.13)$$

$$\underline{y}_p^{\alpha} = \underline{\gamma}, \qquad \overline{y}_p^{\alpha} = \overline{\gamma}, \qquad for\ 0 \le p \le N$$

Formulae for solving Fuzzy initial value problem by using non-linear Trapezoidal formulae based on variety of means are given as follows:

Geometric Mean:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_p^{\alpha} + h\left[\sqrt{\underline{f}^{\alpha}(t_p, y_p)\underline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_p^{\alpha} + h\left[\sqrt{\overline{f}^{\alpha}(t_p, y_p)\overline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right] \quad (4.14)$$

Harmonic Mean:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_p^{\alpha} + 2h\left[\frac{\underline{f}^{\alpha}(t_p, y_p)\underline{f}^{\alpha}(t_{p+1}, y_{p+1})}{\underline{f}^{\alpha}(t_p, y_p) + \underline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_p^{\alpha} + 2h\left[\frac{\overline{f}^{\alpha}(t_p, y_p)\overline{f}^{\alpha}(t_{p+1}, y_{p+1})}{\overline{f}^{\alpha}(t_p, y_p) + \overline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right] \quad (4.15)$$

Heronian Mean:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_p^{\alpha} + \frac{h}{3}\left[\underline{f}^{\alpha}(t_p, y_p) + \underline{f}^{\alpha}(t_{p+1}, y_{p+1}) + \sqrt{\underline{f}^{\alpha}(t_p, y_p)\underline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_{p}^{\alpha} + \frac{h}{3}\left[\overline{f}^{\alpha}(t_p, y_p) + \overline{f}^{\alpha}(t_{p+1}, y_{p+1}) + \sqrt{\overline{f}^{\alpha}(t_p, y_p)\overline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right] \quad (4.16)$$

Contra-hormonic Mean:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_{p}^{\alpha} + h\left[\frac{\underline{f}^{\alpha 2}(t_p, y_p) + \underline{f}^{\alpha 2}(t_{p+1}, y_{p+1})}{\underline{f}^{\alpha}(t_p, y_p) + \underline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_{p}^{\alpha} + h\left[\frac{\overline{f}^{\alpha 2}(t_p, y_p) + \overline{f}^{\alpha 2}(t_{p+1}, y_{p+1})}{\overline{f}^{\alpha}(t_p, y_p) + \overline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

(4.17)

Root Mean Square:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_{p}^{\alpha} + h\left[\sqrt{\frac{\underline{f}^{\alpha 2}(t_p, y_p) + \underline{f}^{\alpha 2}(t_{p+1}, y_{p+1})}{2}}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_{p}^{\alpha} + h\left[\sqrt{\frac{\overline{f}^{\alpha 2}(t_p, y_p) + \overline{f}^{\alpha 2}(t_{p+1}, y_{p+1})}{2}}\right]$$

(4.18)

Centroidal Mean:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_{p}^{\alpha} + \frac{2h}{3}\left[\frac{\underline{f}^{\alpha 2}(t_p, y_p) + \underline{f}^{\alpha}(t_p, y_p)\underline{f}^{\alpha}(t_{p+1}, y_{p+1}) + \underline{f}^{\alpha 2}(t_{p+1}, y_{p+1})}{\underline{f}^{\alpha}(t_p, y_p) + \underline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_{p}^{\alpha} + \frac{2h}{3}\left[\frac{\overline{f}^{\alpha 2}(t_p, y_p) + \overline{f}^{\alpha}(t_p, y_p)\overline{f}^{\alpha}(t_{p+1}, y_{p+1}) + \overline{f}^{\alpha 2}(t_{p+1}, y_{p+1})}{\overline{f}^{\alpha}(t_p, y_p) + \overline{f}^{\alpha}(t_{p+1}, y_{p+1})}\right]$$

(4.19)

Logrithmic Mean:

$$\underline{y}_{p+1}^{\alpha} = \underline{y}_{p}^{\alpha} + h\left[\frac{\underline{f}^{\alpha}(t_{p+1}, y_{p+1}) - \underline{f}^{\alpha}(t_p, y_p)}{ln\left(\frac{\underline{f}^{\alpha}(t_{p+1}, y_{p+1})}{\underline{f}^{\alpha}(t_p, y_p)}\right)}\right]$$

$$\overline{y}_{p+1}^{\alpha} = \overline{y}_{p}^{\alpha} + h\left[\frac{\overline{f}^{\alpha}(t_{p+1}, y_{p+1}) - \overline{f}^{\alpha}(t_p, y_p)}{ln\left(\frac{\overline{f}^{\alpha}(t_{p+1}, y_{p+1})}{\overline{f}^{\alpha}(t_p, y_p)}\right)}\right]$$

(4.20)

4.2 Convergence and Stability

Suppose the exact solution $\left(\underline{Y}(t;\alpha), \overline{Y}(t;\alpha)\right)$ is approximated by some$\left(\underline{y}(t;\alpha),\ \overline{y}(t;\alpha)\right)$. The exact and approximate solutions at $t_{n,}\ 0 \le n \le N$, are denoted by $[Y_n]^{\alpha} = \left[\underline{Y}_n^{\alpha},\ \overline{Y}_n^{\alpha}\right]$ and $[y_n]^{\alpha} = \left[\underline{y}_n^{\alpha},\ \overline{y}_n^{\alpha}\right]$ respectively. Our next result determines the point wise convergence of the Trapezoidal approximates to the exact solution. The following lemma will be applied to show convergence of these approximates; that is,

$\lim_{h\to 0^-} \underline{y}(t;h;\alpha) = \underline{Y}(t;\alpha)$,

$\lim_{h\to 0} \overline{y}(t;h;\alpha) = \overline{Y}(t;\alpha)$

(4.21)

Lemma 4.1

Let a sequence of numbers$\{\omega_n\}_n^N = 0$ satisfy

$|\omega_{n+1}| \leq A|\omega_n| + B, \quad 0 \leq n \leq N-1$,

(4.22)

for some given positive constant A and B. then

$|\omega_n| \leq A^n|\omega_0| + B\frac{A^n-1}{A-1}$, $0 \leq n \leq N-1$,

(4.23)

Let $F(t,u,v)$ and $G(t,u,v)$be the functions F and G, where *u* and *v* are constants and $u \leq v$. The domain where F and G are defined is therefore

$K = \{(t,u,v)|t_0 \leq t \leq T,\ -\infty < v < \infty,\ -\infty < u \leq v\}$.

(4.24)

Theorem 4.1

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^2(K)$, and let the partial derivatives of F, G be bounded over K. Then for arbitrary fixed $\alpha : 0 \leq \alpha \leq 1$, the non-linear Trapezoidal formulae approximates converges to the exact solutions $\underline{Y}(t;\alpha), \overline{Y}(t;\alpha)$ uniformly in t, for $\underline{Y}, \overline{Y} \epsilon\, C^3[t_0,T]$.

Proof. Similar proof [see [15]]

5. Numerical Example:

Example 5.1: Consider the initial value problem $y'(t) = -y(t)$, $y(0) = y_0$,

(5.1)

where $y_0 = [0.96 + 0.04\alpha, 1.01 - 0.01\alpha]$. The exact solution is given by,

$\overline{Y}(0.1;\alpha) = (0.985 + 0.015\alpha)e^{-0.1} - (1-\alpha)0.025e^{0.1}$

(5.2)

$$\overline{Y}(0.1;\alpha) = (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)0.025e^{0.1}$$

The following table 5.1 shows the approximated, exact and absolute error values of example 5.1 by using non-linearTrapezoidal formulae based on variety of Means such as Arithmetic Mean (AM), Logrithmic Mean (LM), Centroidal Mean (CeM), Contra-harmonic Mean (CoM), Harmonic Mean (HM), Heronian Mean (HeM), Geometric Mean (GM), Root Mean Square (RMS) for r = 1 and t = 1.

Table 5.1: Numerical values of example 5.1 at r = 1 and t = 1 for non-linear trapezoidal formulae

Based on	y1	y2	Absolute error 1	Absolute error 2
AM	0.367557	0.367557	3.22E-04	3.22E-04
LM	0.367879	0.367879	1.72E-08	1.72E-08
CeM	0.367235	0.367235	6.44E-04	6.44E-04
CoM	0.366593	0.366593	1.29E-03	1.29E-03
HaM	0.368524	0.368524	6.45E-04	6.45E-04
HeM	0.723951	0.723951	3.56E-01	3.56E-01
GM	2.481012	2.481012	2.11E+00	2.11E+00
RMS	2.486402	2.486402	2.12E+00	2.12E+00
Exac	7879	7879		

The following figures 1 to 8 are the graphical representations of approximate and exact solutions calculated by non-linear Trapezoidal formulae based on variety of means for example 5.1

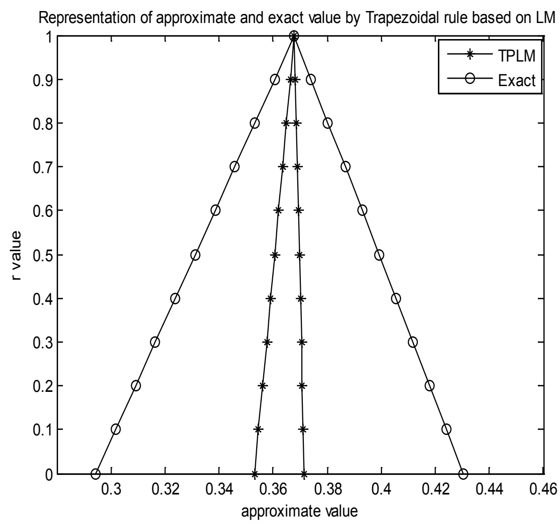


Figure 1 (h=0.1 and t=1) **Figure 2 (h=0.1 and t=1)**

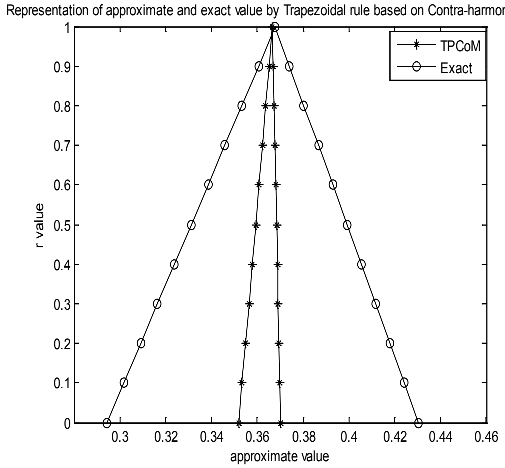


Figure 3 (h=0.1 and t=1)

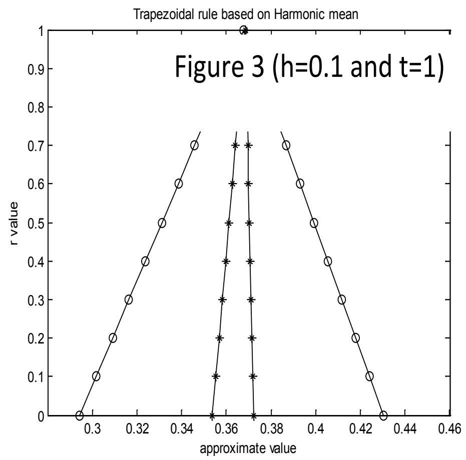


Figure 4 (h=0.1 and t=1)

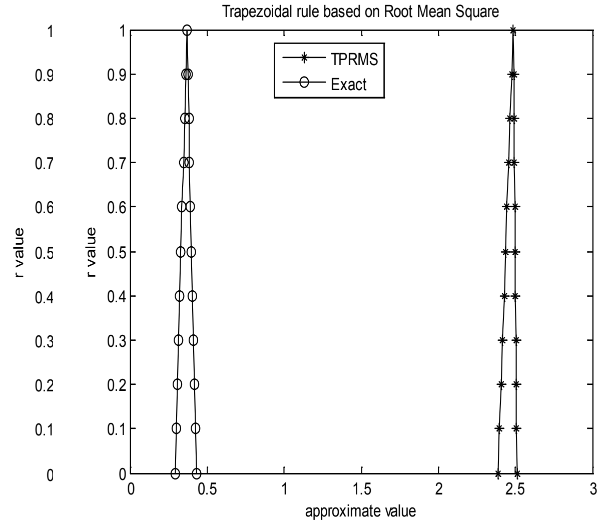


Figure 7 (h=0.1 and t=1)

Figure 8 (h=0.1 and

t=1)

5. Conclusion

The non-linear Trapezoidal formulae based on variety of means for numerical solution of first-order fuzzy differential equations are presented. Also convergence

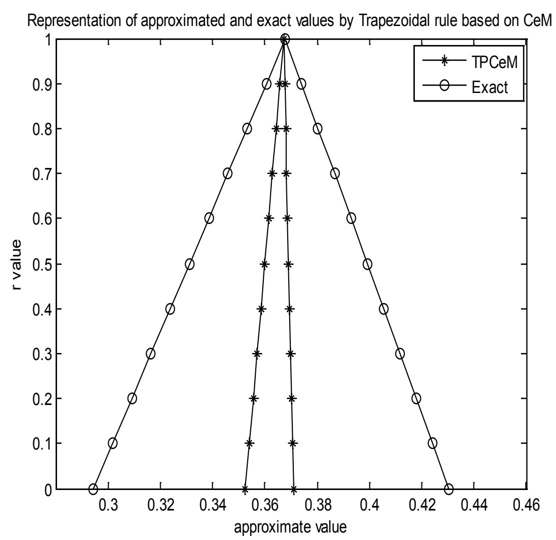
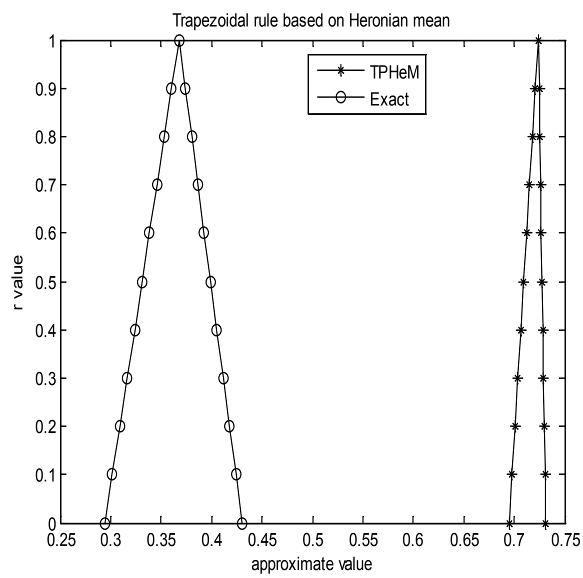


Figure 5 (h=0.1 and t=1)



Figure

6 (h=0.1 and t=1)

and stability of the methods are studied. To illustrate the efficiency of the new methods, a Fuzzy initial value problem is taken and the numerical solution of non-linear trapezoidal formulae based on variety of means are compared with the exact solution and listed in table 5.1 for r = 1 and t = 1. For r = 0 to 1 at time t=1, from figures 1 to 8 it is shown that how the numerical values obtained from the variety of means differ to the exact solution. For the discussed fuzzy initial value problem, the non-linear trapezoidal formula based on LM gives better result when compared to the non-linear trapezoidal formula based on the other means.

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