Research Paper Mathematics

INDICATRICES OF NULL CURVES IN L³

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ABSTRACT

In this paper, we investigate curvature and torsion of indicatrices of null Cartan curves in Lorentz-Minkowski 3-space L3 and give some result on these functions. .

KEYWORDS indicatrix curve, torsion

INTRODUCTION

In Euclidean case, a space curve has a Frenet frame $\{t, n, b\}$ at each point which describes its geometry. Since this frame orthonormal, each element of this set derives a new curve that lies on the sphere of center origin with radius 1. These curves are called indicatrices of the space curve.

There are three categories of curves, according to the causal character of their velocity vector in Lorentz-Minkowski 3-space. Non-null curves have similar properties in Euclidean case. However, null curves have strange frames.

In this paper, we study curvature and torsion of indicatrices of null curves which lie on nullcone and pseudosphere.

PRELIMINARIES

Definition 1. The Lorentz-Minkowski space is the metric space $L^3=(R^3,<,>)$

$$<u,v> = u_1v_1 + u_2v_2 - u_3v_3 , \qquad (1)$$

where $u=(u_1, u_2, u_3)$, $v=(v_1, v_2, v_3) \in R^3$. The metric $<,>$ is called the Lorentzian metric[1]

Definition 2. A vector $v \in L^3$ is said spacelike, timelike or null (lightlike) if $\langle v, v\rangle > 0$ or $v = 0$, $\langle v, v\rangle < 0$, $\langle v, v\rangle = 0$, and $v \neq 0$ respectively. If $v = (v_1, v_2, v_3)$ is in L^3, we define the norm of v by

$$\|v\| = \sqrt{|\langle v, v\rangle|} \qquad (2)$$

[1].

Definition 3. The vectors u and v in L^3 are said to be orthogonal if $\langle u, v\rangle = 0$. A vector u in L^3 which satisfy $\langle u, u\rangle = \pm 1$ is called a unit vector[1].

Definition 4. The nullcone in L^3 is the set of all null vectors of L^3

$$\Lambda_1^2 = \{v \in L^3 : \langle v, v\rangle = 0\} \qquad (3)$$

and the unit pseudo–sphere of center origin with radius 1 defined by

$$S_1^2 = \{v \in L^3 : \langle v, v\rangle = 1\} \qquad (4)$$

[1].

Definition 5. An arbitrary curve $\alpha : I \to L^3$ can locally be a spacelike (resp. timelike or null) if all of its velocity vectors $\alpha'(s)$ are spacelike (resp. timelike, null) vector. A non-null curve $\alpha(s)$ is said to be parametrized by pseudo arc lenght parameter s, if hold $\langle \alpha'(s), \alpha'(s)\rangle = \pm 1$. In this case, the curve α said to be of unit speed[1].

There exists a basis of L^3 for each point of a curve whose variation describes the geometry of the curve. This will be given by the Frenet trihedron {t(s),n(s),b(s)}[1].

Definition 6. Timelike curves and spacelike curves with spacelike or timelike normal vector are called Frenet curves. In this case, the Frenet Equations write in a unified way. If $<t,t>=\varepsilon$ and $<n,n>=\delta$, then

$$t^{'} = \kappa n$$

$$n^{'} = -\delta\kappa t + \tau b \qquad (5)$$

$$b^{'} = \varepsilon\tau n$$

For spacelike curves with null normal vector or null curves, the Frenet equations write as follows: Let $<t,t>=\varepsilon$ and $<n,n>=\delta$ where $\delta, \varepsilon \in \{0,1\}$ and $\delta \neq \varepsilon$. Then,

$$t^{'} = n$$

$$n^{'} = \delta\tau t + \varepsilon\tau n + \delta b \qquad (6)$$

$$b^{'} = \varepsilon t + \delta\tau n - \varepsilon\tau n$$

The torsion is

$$\tau(s) = -\varepsilon\delta < n^{'}(s), b(s) >. \qquad (7)$$

Moreover, as in Euclidean space, one can find the formula for the curvature and torsion functions in the case that the curve is not parametrized by arc length. For Frenet curves

$$\kappa(t) = \frac{\|\alpha^{'}(t)\times\alpha^{''}(t)\|}{\|\alpha^{'}(t)\|^{3}} \qquad (8)$$

$$\tau(t) = -\varepsilon\delta\frac{\langle\alpha^{'}(t)\times\alpha^{''}(t),\alpha^{'''}(t)\rangle}{\|\alpha'(t)\times\alpha''(t)\|^{2}} \qquad (9)$$

[1].

Definition 7. Let $\gamma: I \to L^3$ be a null curve in Lorentz-Minkowski 3-space. Then with respect to an affine parameter, say, p and and a positive oriented set $\{\gamma^{'}(p), \gamma^{''}(p), \gamma^{'''}(p)\}, \forall p \in I$, there exists a local frame $F = \{L, N, W\}$, called Cartan Frame satisfying which satisfy

$$\langle L, L\rangle = \langle N, N\rangle = \langle L, W\rangle = \langle N, W\rangle = 0 \qquad (10)$$

$$\langle L, N\rangle = \langle W, W\rangle = 1 \qquad (11)$$

with the vector product × given by

$$L \times W = -L, L \times N = -W, W \times N = -N. \quad (12)$$

Furthermore, the Cartan equations are given by

$$\begin{array}{rcl} L' & = & W \\ N' & = & \sigma W \\ W' & = & -\sigma L - N \end{array} \qquad (13)$$

where σ is called Cartan curvature of γ with respect to F[2].

Definition 8. A null curve is called a helix if it has constant Cartan curvatures.

INDICATRICES OF NULL CURVES

From (3),(4) and, since

$$\langle L, L\rangle = \langle N, N\rangle = 0, \langle W, W\rangle = 1,$$

we have indicatrices curve $(L), (N)$ and (W) which lie on null cone and pseudosphere, respectively. Furthermore, using Cartan equations, the indicatrices curves (L) and (N) are spacelike curves , the indicatrix curve (W) is spacelike curve (resp. timelike or null), if $\sigma > 0$ (resp. $\sigma<0$, $\sigma=0$).

Then we have following

Proposition 1. Let γ be a null curve in Lorentz-Minkowski 3-space L^3. Then the curvatures and torsion of the indicatrices curve of the null curve γ are

$$\kappa_{(L)} = \sqrt{|-2\sigma|}, \tau_{(L)} = \frac{-\delta_{(L)}\sigma'}{2\sigma},$$

$$\kappa_{(N)} = \sqrt{\left|\frac{-2}{\sigma}\right|}, \tau_{(N)} = \frac{-\delta_{(N)}\sigma'}{2\sigma^2}$$

$$\kappa_{(W)} = \sqrt{\left|\frac{(\sigma')^2 - 8\sigma^3}{8\sigma^3}\right|},$$

$$\tau_{(W)} = \delta_{(W)}\varepsilon_{(W)}\left(\frac{2\sigma\sigma'' - 3(\sigma')^2}{(\sigma')^2 - 8\sigma^3}\right)$$

where $\delta_{(L)} = \langle n_{(L)}, n_{(L)}\rangle$, $\delta_{(N)} = \langle n_{(N)}, n_{(N)}\rangle$, $\delta_{(W)} = \langle n_{(W)}, n_{(W)}\rangle$, $\varepsilon_{(W)} = \langle t_{(W)}, t_{(W)}\rangle$.

Proof : Since $L' = W$, from (13) we have

$$L'' = -\sigma L - N \quad (14)$$

and

$$L''' = -\sigma' L - 2\sigma W. \quad (15)$$

Then, using (12) we have

$$\begin{aligned} L' \times L'' &= W \times (-\sigma L - N) \\ &= -\sigma W \times L - W \times N \\ &= -\sigma L + N \end{aligned}$$

and using (10) and (11)

$$\begin{aligned} \langle L' \times L'', L''' \rangle &= \langle -\sigma L + N, -\sigma' L - 2\sigma W\rangle \\ &= \sigma\sigma'\langle L, L\rangle + 2\sigma^2\langle L, W\rangle - \sigma'\langle N, L\rangle \\ &\quad -2\sigma\langle N, W\rangle \\ &= -\sigma'. \end{aligned}$$

Hence the norm of L' and $L' \times L''$ are

$$\|L'\| = \sqrt{|\langle L', L'\rangle|} = 1$$

and

$$\begin{aligned} \|L' \times L''\| &= \sqrt{|\langle -\sigma L + N, -\sigma L + N\rangle|} \\ &= \sqrt{|-2\sigma|} \end{aligned}$$

respectively. Finally, using (8) and (9) we have the curvature $\kappa_{(L)}$ and the torsion $\tau_{(L)}$ of the indicatrix curve (L) are

$$\kappa_{(L)} = \frac{\|L' \times L''\|}{\|L'\|^3} = \sqrt{|-2\sigma|}$$

and

$$\begin{aligned} \tau_{(L)} &= -\delta_{(L)}\frac{\langle L' \times L'', L'''\rangle}{\|L' \times L''\|^2} \\ &= -\delta_{(L)}\frac{\sigma'}{2\sigma} \end{aligned}$$

where $\delta_{(L)} = \langle n_{(L)}, n_{(L)}\rangle$.

The curvatures and torsions of the indicatrices (N) and (W) have similar above.

CONCLUSIONS

Corollary 1. Let γ be a null curve in Lorentz-Minkowski 3-space L^3. Then, there exist relations as below between the curvatures and the torsions of the indicatrices curve (L) and (N):

$$\kappa_{(L)}.\kappa_{(N)} = 2$$

and

$$\delta_{(N)}\tau_{(L)=}\delta_{(L)}\sigma\tau_{(N)}$$

where $\delta_{(L)} = \langle n_{(L)}, n_{(L)}\rangle$, $\delta_{(N)} = \langle n_{(N)}, n_{(N)}\rangle$.

Corallary 2. Let γ be a null helix in Lorentz-Minkowski 3-space L^3. Then the indicatrices curve (L), (N) and (W) are plane curves.

REFERENCES

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